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1 author:

Mustafa Özdemir
Akdeniz University

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Mustafa Özdemir†

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Abstract

Elliptical rotation is the motion of a point on an ellipse through some angle about a vector. The purpose of this paper is to examine the generation of elliptical rotations and to interpret the motion of a point on an elipsoid using elliptic quaternions, elliptic inner product and elliptic vector product. In this paper, we define elliptic quaternions and generate an elliptical rotation matrix using those quaternions.
Keywords : Elliptic Quaternion, Rotation Matrix, Elliptical Inner and Vector Product.

1 Introduction

A rotation is an example of an isometry, a map that moves points without changing the distances between them. A three dimensional rotation is a linear transformation that describes the motion of a rigid body around an axis and can be expressed with an orthonormal matrix which is called a rotation matrix. 3×3 rotation matrices form a special non abelian orthogonal group, denoted by $\mathbf{SO}(3)$. The group of 3×3 rotation matrices is isomorphic to the group of rotations in a 3 dimensional space. This means that multiplication of rotation matrices corresponds to composition of rotations. Rotation matrices are used extensively for computations in geometry, kinematics, physics, computer graphics, animations, and optimization problems involving the estimation of rigid body transformations. For this reason, the generation of a rotation matrix is considered to be an important problem in mathematics.

There are various representations for rotations as orthonormal matrices, Euler angles, Cayley map, Rodrigues rotation formula, Householder transformation and unit quaternions in the Euclidean space. But to use the unit quaternions is a more useful, natural, and elegant way to perceive rotations compared to other methods. Quaternions were discovered by Sir William R. Hamilton in 1843 and the theory of quaternions was expanded to include applications such as rotations in the early 20th century. The most important property of the quaternions is that every unit quaternion represents a rotation and this plays an important role in the study of rotations in 3-dimensional vector spaces. Quaternions are used especially in computer vision, computer graphics, animation, and kinematics.

A similar relation to the relationship between quaternions and rotations in the Euclidean space exists between split quaternions and rotations in the Minkowski 3-space. Split quaternions are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Besides, the vector part of split

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†Department of Mathematics, Akdeniz University, Antalya, TURKEY, e-mail: mozdemir@akdeniz.edu.tr

quaternions was identified with the Minkowski 3-space [2] . Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector products.
Each of quaternion algebra $\mathbb{H}$ and split quaternion algebra $\widehat{\mathbb{H}}$ is an associative, non-commutative ring with generating by four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Unlike the quaternion algebra, the split-quaternions contain zero divisors and nilpotent elements. We can express any quaternion q as $q = (q_1, q_2, q_3, q_4) = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ or $q = S_q + \mathbf{V}q$ where the symbols $S_q = q_1$ and $\mathbf{V}_q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ denote the scalar and vector parts of q, respectively. If $S_q = 0$ then q is called a pure quaternion. The conjugate of q is denoted by $\overline{q}$, and defined as $\overline{q} = S_q - \mathbf{V}_q$.
A comparison of the properties of quaternions and split quaternions can be given as follows. In this table, only different properties are compared.

Properties	**Quaternion Algebra** $\mathbb{H}$	**Split Quaternion Algebra** $\widehat{\mathbb{H}}$
Algebra	$i^2 = j^2 = k^2 = ijk = -1$	$i^2 = -1, j^2 = k^2 = ijk = 1$
Norm	$\Vert q \Vert = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$	$\sqrt{\lvert q\overline{q}\rvert} = \sqrt{\lvert \overline{q}q\rvert} = \sqrt{\lvert q_1^2 + q_2^2 - q_3^2 - q_4^2\rvert}$
Types	No different types since $q_0^2 + q_1^2 + q_2^2 + q_3^2 > 0$.	Timelike if $q_1^2 + q_2^2 - q_3^2 - q_4^2 > 0$ Spacellike if $q_1^2 + q_2^2 - q_3^2 - q_4^2 < 0\}$ Lightlike if $q_1^2 + q_2^2 - q_3^2 - q_4^2 = 0$
Zero divisor	No zero divisor (Division Ring)	Contains zero divisor (Non-divison ring)
Isomorphic to	an even sub algebra $C\ell_{3,0}^{+}$ with $\{1, e_2e_3 \to j, e_1e_3 \to k, e_1e_2 \to i\}$	an even subalgebra $C\ell_{2,1}^{+}$ with $\{1, e_2e_3 \to i, e_3e_1 \to k, e_1e_2 \to j\}$
Quaternion Product (pq)	$p_1q_1 - \langle \mathbf{V}_p, \mathbf{V}_q \rangle_{\mathbb{E}} + p_1\mathbf{V}_q + q_1\mathbf{V}_p + \mathbf{V}_p \times_{\mathbb{E}} \mathbf{V}_q$ $pq = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$ with Euclidean dot and vector product.	$p_1q_1 + \langle \mathbf{V}_p, \mathbf{V}_q \rangle_{\mathbb{L}} + p_1\mathbf{V}_q + q_1\mathbf{V}_p + \mathbf{V}_p \times_{\mathbb{L}} \mathbf{V}_q$ $pq = \begin{bmatrix} p_1 & -p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & -p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$ with Lorentzian dot and vector product.
Rotation Kind	Euclidean spherical rotations	Lorentzian, hyperbolical and spherical rotations
Unit Quaternion	$q_0 = \dfrac{q}{\Vert q \Vert}$, if $\Vert q \Vert \neq 0$.	$q_0 = \dfrac{q}{\Vert q \Vert}$, if $q_1^2 + q_2^2 - q_3^2 - q_4^2 \neq 0$.
Rotation Group	The set of unit quaternions is a group Each unit quaternion represents a rotation in Euclidean 3-space.	The set of unit split quaternions is not a group. But, The set of timelike quaternions is a group. Each unit timelike quaternion represents a rotation in Minkowski 3-Space.
Polar Form for unit q	$q_0 = (\cos\theta + \varepsilon \sin\theta),\ \varepsilon^2 = -1$	$q = \sinh\theta + \varepsilon_0 \cosh\theta, \varepsilon^2 = 1$ if q spacelike. $q = \cosh\theta + \varepsilon_0 \sinh\theta, \varepsilon^2 = 1$ if q timelike with spacelike $\mathbf{V}_q$ $q = \cos\theta + \varepsilon_0 \sin\theta,\ \varepsilon^2 = -1$ if q timelike with timelike $\mathbf{V}_q$

Each unit quaternion represents a rotation in the Euclidean 3-space. That is, only four numbers are enough to construct a rotation matrix, the only constraint being that the norm of the quaternion is equal to 1. Also, in this method, the rotation angle and the rotation axis can be determined easily. However, this method is only valid in the three dimensional spaces ([15], [18]). In the Lorentzian space, timelike split quaternions are used instead of ordinary usual quaternions ([4], [5]).

Let q and r be two quaternions. Then, the linear transformation $R_q : \mathbb{H} \to \mathbb{H}$ defined by $R_q(r) = qrq^{-1}$ is a quaternion that has the same norm and scalar as r. Since the scalar part of the quaternion r doesn't change under R_q, we will only examine how its vector part $\mathbf{V}_r$ changes under the transformation R_q. We can interpret the rotation of a vector in the Euclidean 3-space using the quaternion product $q\mathbf{V}_r q^{-1}$.

3D Rotation Matrices and Unit Quaternions

If $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ **is a unit quaternion**, then, using the linear transformation $R_q(\mathbf{V}_r) = q\mathbf{V}_r q^{-1}$, the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_1q_2 + 2q_3q_0 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_1q_0 \\ 2q_1q_3 - 2q_2q_0 & 2q_1q_0 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{4}$$

This rotation matrix represents a rotation through angle 2θ about the axis $\boldsymbol{\varepsilon} = (q_1, q_2, q_3)$. In the Lorentzian space, the rotation matrix corresponding to a **unit timelike quaternion** $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is,

$$R_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix}. \text{ (see[4]).}$$

Details about generating rotation matrices, particularly in the Euclidean and Lorentzian spaces, using these methods can be found in various papers, some of which are given in the reference section. Those authors mostly studied the rotation matrices in the positive definite scalar product space whose associated matrices are $\mathrm{diag}(\pm 1, \cdots, \pm 1)$, and interpreted the results geometrically. For example, quaternions and timelike split quaternions were used to generate rotation matrices in the three dimensional Euclidean and Lorentzian spaces where the associated matrices were $diag(1,1,1)$ and $diag(-1,1,1)$, respectively. In these spaces, rotations occur on the sphere $x^2+y^2+z^2 = r^2$ or the hyperboloids $-x^2+y^2+z^2 = \pm r^2$. That is, Euclidean and Lorentzian rotation matrices help us to understand spherical and hyperbolic rotations. In the Euclidean space, a rotation matrix rotates a point or a rigid body through a circular angle about an axis. That is, the motion happens on a circle. Similarly, in the Lorentzian space, a rotation matrix rotates a point through an angle about an axis circularly or hyperbolically depending on whether the rotation axis is timelike or spacelike, respectively.

In this paper, we investigate elliptical rotation matrices, which are orthogonal matrices in the scalar product space, whose associated matrix is $\mathrm{diag}(a_1, a_2, a_3)$ with $a_1, a_2, a_3 \in \mathbb{R}^+$. First, we choose a proper scalar product to the given ellipse (or ellipsoid) such that this ellipse (or ellipsoid) is equivalent to a circle (or sphere) for the scalar product space. That is, the scalar product doesn't change the distance between any point on the ellipse (or ellipsoid) and origin. Interpreting a motion on an ellipsoid is an important concept since planets usually have ellipsoidal shapes and elliptical orbits. The geometry of ellipsoid can be examined using affine transformations, because of an ellipsoid can be considered as an affine map of the unit sphere.

The aim of this study is to explain the motion on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

as a rotation, using the proper inner product, vector product and elliptical orthogonal matrices. In this method, the elliptical inner product, the vector product and the angles are compatible with the parameters θ and β of the parametrization

$$\varphi(\theta,\beta)=(a\cos\theta\cos v, b\cos\theta\sin\beta, c\sin\theta).$$

In this paper, we defined the elliptic quaternions and generate elliptical rotations using unit elliptic quaternions for a given ellipsoid.

2 Elliptical Inner and Vector Product

We begin with a brief review of scalar products. More informations can be found in ([8], [9] and, [14]). Consider the map

$$\mathcal{B}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\quad (\mathbf{u},\mathbf{v})\to\mathcal{B}(\mathbf{u},\mathbf{v})$$

for $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$. If such a map is linear in each argument, that is,

$$\begin{aligned}\mathcal{B}(a\mathbf{u}+b\mathbf{v},\mathbf{w})&=a\mathcal{B}(\mathbf{u},\mathbf{w})+b\mathcal{B}(\mathbf{v},\mathbf{w}),\\ \mathcal{B}(\mathbf{u},c\mathbf{v}+d\mathbf{w})&=c\mathcal{B}(\mathbf{u},\mathbf{v})+d\mathcal{B}(\mathbf{u},\mathbf{w}),\end{aligned}$$

where, $a,b,c,d\in\mathbb{R}$ and $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$, then it is called a bilinear form. Given a bilinear form on $\mathbb{R}^n$, there exists a unique $\Omega\in\mathbb{R}^{n\times n}$ square matrix such that for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, $\mathcal{B}(\mathbf{u},\mathbf{v})=\mathbf{u}^t\Omega\mathbf{v}$. Ω is called "*the matrix associated with the form*" with respect to the standard basis and we will denote $\mathcal{B}(\mathbf{u},\mathbf{v})$ as $\mathcal{B}_\Omega(\mathbf{u},\mathbf{v})$ as needed. A bilinear form is said to be symmetric or skew symmetric if $\mathcal{B}(\mathbf{u},\mathbf{v})=\mathcal{B}(\mathbf{v},\mathbf{u})$ or $\mathcal{B}(\mathbf{u},\mathbf{v})=-\mathcal{B}(\mathbf{v},\mathbf{u})$, respectively. Hence, the matrix associated with a symmetric bilinear form is symmetric, and similarly, the associated matrix of a skew symmetric bilinear form is skew symmetric. Also, a bilinear form is nondegenerate if its associated matrix is non-singular. That is, for all $\mathbf{u}\in\mathbb{R}^n$, there exists $\mathbf{v}\in\mathbb{R}^n$, such that $\mathcal{B}(\mathbf{u},\mathbf{v})\neq 0$. A real scalar product is a non-degenerate bilinear form. The space $\mathbb{R}^n$ equipped with a fixed scalar product is said to be a real scalar product space. Also, some scalar products, like the dot product, have positive definitely property. That is, $\mathcal{B}(\mathbf{u},\mathbf{u})\geq 0$ and $\mathcal{B}(\mathbf{u},\mathbf{u})=0$ if and only if $\mathbf{u}=0$. Now, we will define a positive definite scalar product, which we call the $\mathcal{B}$-inner product or elliptical inner product.

Let $\mathbf{u}=(u_1,u_2,...,u_n)$, $\mathbf{w}=(w_1,w_2,...,w_n)\in\mathbb{R}^n$ and $a_1,a_2,..,a_n\in\mathbb{R}^+$. Then the map

$$\mathcal{B}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\quad \mathcal{B}(\mathbf{u},\mathbf{w})=a_1u_1w_1+a_2u_2w_2+\cdots+a_nu_nw_n$$

is a positive definite scalar product. We call it elliptical inner product or $\mathcal{B}$-inner product. The real vector space $\mathbb{R}^n$ equipped with the elliptical inner product will be represented by $\mathbb{R}^n_{a_1,a_2,...,a_n}$ or $\mathbb{R}^n_{\mathcal{B}}$. Note that the scalar product $\mathcal{B}(\mathbf{u},\mathbf{v})$ can be written as $\mathcal{B}(\mathbf{u},\mathbf{w})=\mathbf{u}^t\Omega\mathbf{w}$ where associated matrix is

$$\Omega=\begin{bmatrix}a_1&0&\cdots&0\\0&a_2&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&a_n\end{bmatrix}. \qquad (1)$$

The number $\sqrt{\det \Omega}$ will be called "*constant of the scalar product*" and denoted by Δ in the rest of the paper. Two vectors $\mathbf{u}$ and $\mathbf{w}$ are called $\mathcal{B}$-orthogonal or elliptically orthogonal vectors if $\mathcal{B}(\mathbf{u},\mathbf{w})=0$. In addition, if their norms are 1, then they are called $\mathcal{B}$-orthonormal vectors. If $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_n\}$ is an $\mathcal{B}$-orthonormal base of $\mathbb{R}^n_{a_1,a_2,...,a_n}$, then $\det(\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_n)=\Delta^{-1}$. The cosine of the angle between two vectors $\mathbf{u}$ and $\mathbf{w}$ is defined as,

$$\mathbf{cos}\theta=\frac{\mathcal{B}(\mathbf{u},\mathbf{w})}{\|\mathbf{u}\|_{\mathcal{B}}\,\|\mathbf{w}\|_{\mathcal{B}}}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid.

Let $\mathcal{B}$ be a non degenerate scalar product, Ω the associated matrix of $\mathcal{B}$, and $R\in\mathbb{R}^{n\times n}$ is any matrix.

i) If $\mathcal{B}(R\mathbf{u},R\mathbf{w})=\mathcal{B}(\mathbf{u},\mathbf{w})$ for all vectors $\mathbf{u},\mathbf{w}\in\mathbb{R}^n$, then R is called a $\mathcal{B}$-orthogonal matrix. It means that orthogonal matrices preserve the norm of vectors and satisfy the matrix equality $R^t\Omega R=\Omega$. Also, all rows (or columns) are $\mathcal{B}$-orthogonal to each other. We denote the set of $\mathcal{B}$-orthogonal matrices by $\mathbf{O}_{\mathcal{B}}(n)$. That is,

$$\mathbf{O}_{\mathcal{B}}(n)=\{R\in\mathbb{R}^{n\times n}:R^t\Omega R=\Omega \text{ and } \det R=\pm1\}.$$

$\mathbf{O}_{\mathcal{B}}(n)$ is a subgroup of $\mathbf{Gl}_{\mathcal{B}}(n)$. It is sometimes called the isometry group of $\mathbb{R}^n$ associated with scalar product $\mathcal{B}$. The determinant of a $\mathcal{B}$-orthogonal matrix can be either -1 or 1. If $\det R=1$, then we call it a $\mathcal{B}$-rotation matrix or an elliptical rotation matrix. If $\det R=-1$, we call it an elliptical reflection matrix. Although the set $\mathbf{O}_{\mathcal{B}}(n)$ is not a linear subspace of $\mathbb{R}^{n\times n}$, it is a Lie group. The isometry group for the bilinear or sesquilinear forms can be found in [8]. The set of the $\mathcal{B}$-rotation matrices of $\mathbb{R}^n$ can be expressed as follows:

$$\mathbf{SO}_{\mathcal{B}}(n)=\{R\in\mathbb{R}^{n\times n}:R^t\Omega R=\Omega \text{ and } \det R=1\}.$$

$\mathbf{SO}_{\mathcal{B}}(n)$ is a subgroup of $\mathbf{O}_{\mathcal{B}}(n)$.

ii) If $\mathcal{B}(S\mathbf{u},\mathbf{w})=\mathcal{B}(\mathbf{u},S\mathbf{w})$ for all vectors $\mathbf{u},\mathbf{w}\in\mathbb{R}^n$, then S is called a $\mathcal{B}$-symmetric matrix. It satisfies $S^t\Omega=\Omega S$. The set of $\mathcal{B}$-symmetric matrices, defined by

$$\mathbb{J}=\left\{S\in\mathbb{R}^{n\times n}:\mathcal{B}(S\mathbf{u},\mathbf{w})=\mathcal{B}(\mathbf{u},S\mathbf{w}) \text{ for all } \mathbf{u},\mathbf{w}\in\mathbb{R}^n\right\}$$

is a Jordan algebra [8]. It is a subspace of the vector space of real $n\times n$ matrices, with dimension $n(n+1)/2$. Any $\mathcal{B}$-symmetric matrix in $\mathbb{R}^n_{a_1,a_2,...,a_n}$ can be defined as

$$S=\left[\frac{\Delta a_{ij}}{a_i}\right]_{n\times n} \tag{2}$$

where $a_{ij}=a_{ji}$ and $a_{ij}\in\mathbb{R}$.

iii) If $\mathcal{B}(T\mathbf{u},\mathbf{w})=-\mathcal{B}(\mathbf{u},T\mathbf{w})$ for all vectors $\mathbf{u},\mathbf{w}\in\mathbb{R}^n$, then T is called a $\mathcal{B}$-skew-symmetric matrix. Also, $T^t\Omega=-\Omega T$. The set of $\mathcal{B}$-skew symmetric matrices, defined by

$$\mathbb{L}=\left\{T\in\mathbb{R}^{n\times n}:\mathcal{B}(T\mathbf{u},\mathbf{w})=-\mathcal{B}(\mathbf{u},T\mathbf{w}) \text{ for all } \mathbf{u},\mathbf{w}\in V\right\}$$

is a Lie algebra [8]. It is a subspace of the vector space of real $n\times n$ matrices, with dimension $n(n-1)/2$, as well. Any $\mathcal{B}$-skew-symmetric matrix in $\mathbb{R}^n_{a_1,a_2,...,a_n}$ can be defined as,

$$T=[t_{ij}]_{n\times n} \quad \text{with} \quad t_{ij}=\begin{cases}\dfrac{\Delta a_{ij}}{a_i} & i>j\\ \dfrac{-\Delta a_{ij}}{a_i} & i<j\\ 0 & i=j\end{cases} \tag{3}$$

where $a_{ij}=a_{ji}$ and $a_{ij}\in\mathbb{R}$.

For example, in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$, the symmetric and skew symmetric matrices are

$$S = \Delta \begin{bmatrix} a_{11}/a_1 & x/a_1 & y/a_1 \\ x/a_2 & a_{22}/a_2 & z/a_2 \\ y/a_3 & z/a_3 & a_{33}/a_3 \end{bmatrix} \quad \text{and} \quad T = \Delta \begin{bmatrix} 0 & x/a_1 & y/a_1 \\ -x/a_2 & 0 & z/a_2 \\ -y/a_3 & -z/a_3 & 0 \end{bmatrix}.$$

Note that, even if we omit the scalar product constant Δ in S or T, they will still be symmetric or skew symmetric matrix, respectively. But then, we cannot generate elliptical rotation matrices using the Rodrigues and Cayley formulas. So, we will keep the constant Δ.

Elliptical Vector Product

Now, we define the elliptical vector product, which is related to elliptical inner product. Let $u_i = (u_{i1}, u_{i2}, ..., u_{in}) \in \mathbb{R}^n$ for $i = 1, 2, ..., n-1$ and $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ be standard unit vectors for $\mathcal{B}$. Then, the elliptical vector product in $\mathbb{R}^n_{a_1,a_2,...,a_n}$ is defined as,

$$\mathbb{R}^n_{a_1,a_2,...,a_n} \times \mathbb{R}^n_{a_1,a_2,...,a_n} \times \cdots \times \mathbb{R}^n_{a_1,a_2,...,a_n} \to \mathbb{R}^n_{a_1,a_2,...,a_n},$$
$$(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n) \to \mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \cdots \times \mathbf{u}_{n-1})$$

$$\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \cdots \times \mathbf{u}_{n-1}) = \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 & \cdots & \mathbf{e}_n/a_n \\ u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ u_{21} & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{(n-1)1} & u_{(n-1)2} & u_{(n-1)3} & \cdots & u_{(n-1)n} \end{bmatrix} \tag{4}$$

The vector $\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \cdots \times \mathbf{u}_{n-1})$ is $\mathcal{B}$-orthogonal to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, ..., \mathbf{u}_{n-1}$ geometrically. As a special case, the 3-dimensional elliptical vector product of $\mathbf{u}_1$ and $\mathbf{u}_2$ is defined as

$$\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2) = \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix}$$

for the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$, where $\Delta = \sqrt{a_1 a_2 a_3}$.

The ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$, $a_i \in \mathbb{R}^+$ is the unit sphere for this space. The end point any unit vector in $\mathbb{R}^3_{a_1,a_2,a_3}$ falls on the ellipsoid. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors in $\mathbb{R}^3_{a_1,a_2,a_3}$, then $\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2)$ is also unit vector and it is elliptically orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. The standard vector product is a special case of the elliptical vector product. That is, if we take $a_1 = a_2 = a_3 = 1$, we end up standard vector product, standard inner product and, standard orthogonality in the three dimensional Euclidean space.

3 3D Elliptical Rotations

Let's take the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ where $a_i \in \mathbb{R}^+$. The scalar product for this ellipsoid is

$$\mathcal{B}(\mathbf{u}, \mathbf{w}) = a_1u_1w_1 + a_2u_2w_2 + a_3u_1w_3,$$

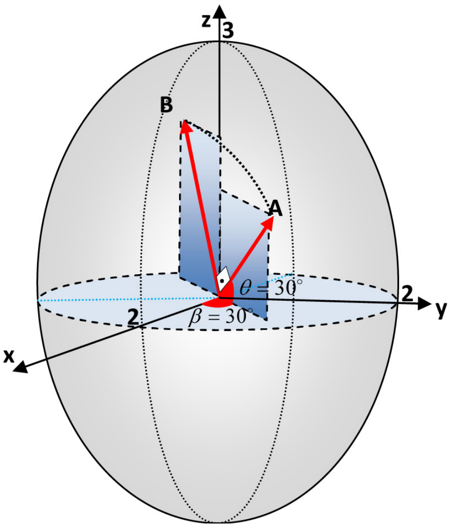
for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Also, the vector product is

$$\mathcal{V}(\mathbf{u}\times\mathbf{v}){=}\Delta\det\begin{bmatrix}\mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}{=}\Delta\begin{bmatrix}0 & -u_3/a_1 & u_2/a_1\\ u_3/a_2 & 0 & -u_1/a_2\\ -u_2/a_3 & u_1/a_3 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}{=}T\left(\mathbf{v}^t\right)$$

where $\Delta=\sqrt{a_1a_2a_3}$. The matrix

$$T=\Delta\begin{bmatrix}0 & -u_3/a_1 & u_2/a_1\\ u_3/a_2 & 0 & -u_1/a_2\\ -u_2/a_3 & u_1/a_3 & 0\end{bmatrix}\tag{5}$$

is skew symmetric in $\mathbb{R}^3_{a_1,a_2,a_3}$. That is, $T^t\Omega=-\Omega T$. So, the vector product in $\mathbb{R}^3_{a_1,a_2,a_3}$ can be viewed as a linear transformation, which corresponds to multiplication by a skew symmetric matrix. The characteristic polynomial of T is, $P(x)=x^3+\|\mathbf{u}\|^2x$ whose eigenvalues are $x_1=0$ and $x_{2,3}=\pm\|\mathbf{u}\|i$. According to characteristic polynomial $T^3+\|\mathbf{u}\|^2T=0$. So, if we take a unit vector $\mathbf{u}\in\mathbb{R}^3_{a_1,a_2,a_3}$, we get $T^3=-T$ and we can use Rodrigues and Cayley formulas. (See the paper "**An Alternative Approach tor elliptical motion**" published online in "Advances in Applied Clifford Algebras.")



Example 1 *A parametrization of the ellipsoid* $\dfrac{x^2}{4}+\dfrac{y^2}{4}+\dfrac{z^2}{9}=1$ *is*

$$\alpha(\theta,\beta)=(2\cos\theta\cos\beta,2\cos\theta\sin\beta,3\sin\theta)$$

where $\theta\in[0,\pi)$ *and* $\beta\in[0,2\pi)$. *Let's take the points*

$$A{=}\alpha(30^\circ,30^\circ)=\left(3/2,\sqrt{3}/2,3/2\right)\quad\textit{and}\quad B{=}\alpha(120^\circ,30^\circ)=\left(-\sqrt{3}/2,-1/2,3\sqrt{3}/2\right)$$

on the ellipsoid. Let's find the rotation matrix which is rotate the point A to B elliptically. We have $a_1=a_2=1/4$ and $a_3=1/9$. So, $\Delta=1/12$. First, using the vector product of $\mathbf{x}=\overrightarrow{OA}$ *and* $\mathbf{y}=\overrightarrow{OB}$ *in* $\mathbb{R}^3_{1/4,1/4,1/9}$*, we find the rotation axis* $\mathbf{u}$.

$$\mathcal{V}(\mathbf{x}\times\mathbf{y})=\frac{1}{12}\begin{vmatrix}4i & 4j & 9k\\ 3/2 & \sqrt{3}/2 & 3/2\\ -\sqrt{3}/2 & -\dfrac{1}{2} & 3\sqrt{3}/2\end{vmatrix}=\left(1,-\sqrt{3},0\right).$$

Since $\mathcal{V}(\mathbf{x}\times\mathbf{y})$ *is unit vector in* $\mathbb{R}^3_{1/4,1/4,1/9}$, *we get* $\mathbf{u}=(1,-\sqrt{3},0)$. *Thus, we can obtain the elliptical rotation matrix*

$$R^{\mathbf{u}}_{\theta}T(\theta)=\frac{1}{12}\begin{bmatrix} 9\cos\theta+3 & 3\sqrt{3}(\cos\theta-1) & -4\sqrt{3}\sin\theta \\ 3\sqrt{3}(\cos\theta-1) & 3\cos\theta+9 & -4\sin\theta \\ 9\sqrt{3}\sin\theta & 9\sin\theta & 12\cos\theta \end{bmatrix}$$

by using Rodrigues rotation formula (see [1]). This matrix describes an elliptical rotation on a great ellipse such that it is intersection of the ellipsoid and the plane passing through the origin and $\mathcal{B}$*-orthogonal to* $\mathbf{u}$. *It can be easily found that equation of the plane is* $x=\sqrt{3}y$. *So,* $R^{\mathbf{u}}_{\theta}$ *represents an elliptical rotation over the the great ellipse is* $y^2+\frac{1}{9}z^2=1$, $y=\sqrt{3}x$. *Also, the elliptical rotation angle is* $\pi/2$, *since* $\cos\theta=\mathcal{B}(\mathbf{x},\mathbf{y})=0$. *Thus, we find*

$$R^{\mathbf{u}}_{\pi/2}=\frac{1}{12}\begin{bmatrix} 3 & -3\sqrt{3} & -4\sqrt{3} \\ -3\sqrt{3} & 9 & -4 \\ 9\sqrt{3} & 9 & 0 \end{bmatrix}. \tag{6}$$

The matrix (6), rotates the point A to the point B elliptically over the great ellipse $\frac{x^2}{2}+\frac{z^2}{9}=1$, $y=x$.

4 Elliptic Quaternions

Elliptic Quaternions for a given ellipsoid

To get an elliptical rotation matrix, first we define the set of elliptic quaternions suitable for the ellipsoid

$$a_1x^2+a_2y^2+a_3z^2=1.$$

Let's take four basic elements $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ satisfying the equalities

$$\mathbf{i}^2=-a_1,\qquad \mathbf{j}^2=-a_2,\qquad \mathbf{k}^2=-a_3$$

and

$$\mathbf{ij}=\frac{\Delta}{a_3}\mathbf{k}=-\mathbf{ji},\qquad \mathbf{jk}=\frac{\Delta}{a_1}\mathbf{i}=-\mathbf{kj},\qquad \mathbf{ki}=\frac{\Delta}{a_2}\mathbf{j}=-\mathbf{ik}$$

where $a_1,a_2,a_3\in\mathbb{R}^+$ and $\Delta=\sqrt{a_1a_2a_3}$.

The set of elliptic quaternions will be denoted by $\mathbb{H}_{a_1,a_2,a_3}$. This set is an associative, non-commutative division ring with our basic elements $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$. If we take $a_1=a_2=a_3=1$, we get the usual quaternion algebra. The elliptic quaternion product table is given below.

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-a_1$	$\Delta\mathbf{k}/a_3$	$-\Delta\mathbf{j}/a_2$
$\mathbf{j}$	$\mathbf{j}$	$-\Delta\mathbf{k}/a_3$	$-a_2$	$\Delta\mathbf{i}/a_1$
$\mathbf{k}$	$\mathbf{k}$	$\Delta\mathbf{j}/a_2$	$-\Delta\mathbf{i}/a_1$	$-a_3$

For each ellipsoid, we define a quaternion product using scalar product and vector product such that the ellipsoid is a unit sphere for this scalar product space. Now, let's define the quaternion product for the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$.

Elliptic Quaternion Product

The elliptic quaternion product of two quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as

$$p_0q_0 - \mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) + p_0\mathbf{V}_q + q_0\mathbf{V}_p + \mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) \tag{7}$$

where

$$\mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) = a_1p_1q_1 + a_2p_2q_2 + a_3p_3q_3,$$

and

$$\mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) = \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} = \Delta \begin{bmatrix} 0 & -p_3/a_1 & p_2/a_1 \\ p_3/a_2 & 0 & -p_1/a_2 \\ -p_2/a_3 & p_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

are the elliptical scalar product and the elliptical vector product for $a_1x^2 + a_2y^2 + a_3z^2 = 1$, respectively. Here, Δ is equal to $\sqrt{a_1a_2a_3}$.

If p and q are pure, then

$$\begin{aligned} pq &= -\mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) + \mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) \\ &= -(a_1p_1q_1 + a_2p_2q_2 + a_3p_3q_3) + \Delta \begin{vmatrix} i/a_1 & j/a_2 & k/a_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}. \end{aligned}$$

The elliptic quaternion product for $\mathbb{H}_{a_1,a_2,a_3}$ can be expressed as $(pq)^T = F_p q^T$ where,

$$F_p(\Delta, a_1, a_2, a_3, p_0, p_1, p_2, p_3) = \begin{bmatrix} p_0 & -a_1p_1 & -a_2p_2 & -a_3p_3 \\ p_1 & p_0 & -\dfrac{p_3\Delta}{a_1} & \dfrac{p_2\Delta}{a_1} \\ p_2 & \dfrac{p_3\Delta}{a_2} & p_0 & -\dfrac{p_1\Delta}{a_2} \\ p_3 & -\dfrac{p_2\Delta}{a_3} & \dfrac{p_1\Delta}{a_3} & p_0 \end{bmatrix} \text{ and } q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

For example, let $p, q \in \mathbb{H}_{2,2,1}$. Then, the elliptic quaternion product of p and q defined is

$$pq = \begin{bmatrix} p_0 & -2p_1 & -2p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -2p_2 & 2p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

For $p = 1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $q = 2 + 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, we get $pq = (-32, 13, 17, -9)$. This can also be calculated using the product table

	1	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	-2	$2\mathbf{k}$	$-\mathbf{j}$
j	**j**	$-2\mathbf{k}$	-2	**i**
k	**k**	**j**	$-\mathbf{i}$	-1

.

Example 2 *Let's define the set of elliptical quaternions and the elliptical quaternion product for the ellipsoid* $(E) : \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{9} = 1$.
Since, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$ *and* $a_3 = \frac{1}{9}$, *we get*

$$\Delta = \sqrt{a_1 a_2 a_3} = \frac{1}{6}.$$

So, the set of elliptical quaternion for (E) *is,*

$$\mathbb{H}_{1/2,1/2,1/9} = \left\{ q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : \mathbf{i}^2 = -\frac{1}{2},\ \mathbf{j}^2 = -\frac{1}{2},\ \mathbf{k}^2 = -\frac{1}{9},\ \mathbf{ijk} = -\frac{1}{6} \right\}.$$

Using the scalar product an vector product

$$\mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) = \frac{1}{2}p_1q_1 + \frac{1}{2}p_2q_2 + \frac{1}{9}p_3q_3$$

and

$$\mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) = \frac{1}{6}\begin{bmatrix} 0 & -2p_3 & 2p_2 \\ 2p_3 & 0 & -2p_1 \\ -9p_2 & 9p_1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

for the ellipsoid $(E) : \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{9} = 1$, *the elliptic quaternion product of the quaternions*

$$p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \text{ and } q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

is defined as

$$pq = \begin{bmatrix} p_0q_0 - \frac{1}{2}p_1q_1 - \frac{1}{2}p_2q_2 - \frac{1}{9}p_3q_3 \\ p_0q_1 + p_1q_0 + \frac{1}{3}p_2q_3 - \frac{1}{3}p_3q_2 \\ p_0q_2 + p_2q_0 - \frac{1}{3}p_1q_3 + \frac{1}{3}p_3q_1 \\ p_0q_3 + \frac{3}{2}p_1q_2 - \frac{3}{2}p_2q_1 + q_0p_3 \end{bmatrix}$$

or

$$(pq)^T = Fq^T$$

where

$$F_p\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{9}, p_0, p_1, p_2, p_3\right) = \begin{bmatrix} p_0 & -p_1/2 & -p_2/2 & -p_3/9 \\ p_1 & p_0 & -p_3/3 & p_2/3 \\ p_2 & p_3/3 & p_0 & -p_1/3 \\ p_3 & -3p_2/2 & 3p_1/2 & p_0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \text{ and } q^T = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

Let's find product of the quaternions $p = 1 + 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ *and* $q = 3 + 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *using the elliptical quaternion product defined for the ellipsoid* $(E) : \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{9} = 1$.

$$F_p\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{9}, 1, 2, 3, 6\right) q^T = \begin{bmatrix} 1 & -1 & -\frac{3}{2} & -\frac{2}{3} \\ 2 & 1 & -2 & 1 \\ 3 & 2 & 1 & -\frac{2}{3} \\ 6 & -\frac{9}{2} & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ 13 \\ 18 \end{bmatrix}.$$

Clifford Algebra and Elliptic Quaternions
Remember that the algebra is formed by a vector space $\mathbb{V}$ equipped with a quadratic form Q with the following equalities

$$\mathbf{v}^2 = Q(\mathbf{v})\,;$$
$$\mathbf{uv} + \mathbf{vu} = 2\mathcal{B}_Q(\mathbf{u}, \mathbf{v})$$

is called a *Clifford algebra* and is denoted by $C\ell(\mathbb{V}, Q)$. If $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is a base for an n-dimensional vector space $\mathbb{V}$, then $C\ell(\mathbb{V}, Q)$ is formed by the multivectors

$$\{1\} \cup \{\mathbf{e}_{i_1}\mathbf{e}_{i_2}...\mathbf{e}_{i_k} : 1 \leq i_1 \leq ... \leq i_k \leq n,\ 1 \leq k \leq n\}$$

with $\dim(C\ell(\mathbb{V}, Q)) = 2^n$. Since the Clifford product of two even multivectors is an even multivector, they define an even subalgebra of $C\ell(\mathbb{V}, Q)$. The even subalgebra of an n-dimensional Clifford algebra is isomorphic to a Clifford algebra of $(n-1)$ dimensions and it is denoted by $C\ell^+(\mathbb{V}, Q)$. The Hamiltonian quaternion algebra $\mathbb{H}$ is isomorphic with the even subalgebra $C\ell^+_{3,0} = C\ell\left(\mathbb{R}^3, Q = x_1^2 + x_2^2 + x_3^2\right)$ by $\{1, e_2e_3 \to \mathbf{j}, e_1e_3 \to \mathbf{k}, e_1e_2 \to \mathbf{i}\}$ and the split quaternion algebra $\widehat{\mathbb{H}}$ is isomorphic with the even subalgebra $C\ell^+_{2,1} = C\ell\left(\mathbb{R}^3_1, Q = -x_1^2 + x_2^2 + x_3^2\right)$ by $\{1, e_2e_3 \to \mathbf{i}, e_3e_1 \to \mathbf{k}, e_1e_2 \to \mathbf{j}\}$ [15]. Similarly, the elliptic quaternion algebra is an even subalgebra of the Clifford algebra

$$C\ell\left(\mathbb{R}^3\right) = \left\{q = q_0 + \mathbf{e}_1q_1 + \mathbf{e}_2q_2 + \mathbf{e}_3q_3 : \mathbf{e}_1^2 = a_1,\ \mathbf{e}_2^2 = a_2,\ \mathbf{e}_3^2 = a_3,\ \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0\right\}$$

associated with the nondegenerate quadratic form $Q(x) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2$ and is denoted by $C\ell^+\left(\mathbb{R}^3, a_1x_1^2 + a_2x_2^2 + a_3x_3^2\right)$, or shortly $C\ell^+\left(\mathbb{R}^3_{a_1,a_2,a_3}\right)$. $\mathbb{H}_{a_1,a_2,a_3}$ is isomorphic to $C\ell^+\left(\mathbb{R}^3_{a_1,a_2,a_3}\right)$ with

$$\{1, \frac{a_1}{\Delta}e_2e_3 \to \mathbf{i}, \frac{a_2}{\Delta}e_1e_3 \to \mathbf{j}, \frac{a_3}{\Delta}e_1e_2 \to \mathbf{k}\}.$$

For the quadratic form $Q = a_1x_1^2 + a_1x_2^2 + a_1x_3^2$, recall that the elliptical inner product can be obtained by using the equality

$$\mathcal{B}_Q(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left[Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})\right].$$

So, we get $\mathcal{B}_Q(\mathbf{x}, \mathbf{y}) = a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3$ for $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Thus, we can construct an elliptic quaternion algebra for any elliptical inner product space.
Conjugate, norm and inverse of an elliptic quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be defined similar to usual quaternions :

$$\begin{aligned}
\overline{q} &= q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}, \\
\|q\| &= \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + a_1q_1^2 + a_2q_2^2 + a_3q_3^2}, \\
q^{-1} &= \frac{\overline{q}}{\|q\|^2}.
\end{aligned}$$

Also, each elliptic quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be written in the form

$$q_0 = \|q\|\left(\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta\right)$$

where

$$\cos\theta = \frac{q_0}{\|q\|} \quad \text{and} \quad \sin\theta = \frac{\sqrt{a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2}}{\|q\|}.$$

Here, $\varepsilon_0 = \dfrac{(q_1, q_2, q_3)}{\sqrt{a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2}}$ is a unit vector in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$ satisfying the equality $\varepsilon_0^2 = -1$. It is called the axis of the rotation. For example, if $q = 1 + 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} \in \mathbb{H}_{2,2,1}$, then $\|q\| = \sqrt{1^2 + 2\cdot 2^2 + 2\cdot 1^2 + 1\cdot 5^2} = 6$ and we can write

$$q = \frac{1}{6} + \frac{\sqrt{35}}{6}\frac{(2,1,5)}{\sqrt{35}} = \cos\theta + \frac{(2,1,5)}{\sqrt{35}}\sin\theta$$

where $\varepsilon_0 = \dfrac{1}{\sqrt{35}}(2,1,5)$ is a unit vector in $\mathbb{R}^3_{2,2,1}$ with $\varepsilon_0^2 = -1$.

Theorem 1 *Each a unit elliptic quaternion represents an elliptical rotation on the an ellipsoid. If*

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\theta + \varepsilon_0 \sin\theta \in \mathbb{H}_{a_1,a_2,a_3}$$

is a unit elliptic quaternion, then the linear map $R_\theta(\mathbf{v}) = q\mathbf{v}q^{-1}$ gives an elliptical rotation through the elliptical angle 2θ, about the axis ε_0, where $\mathbf{v} \in \mathbb{R}^3$. The elliptical rotation matrix to corresponding to the quaternion q is

$$R_\theta^q = \begin{bmatrix} q_0^2 + a_1 q_1^2 - a_2 q_2^2 - a_3 q_3^2 & 2a_2 q_1 q_2 - 2\dfrac{q_0 q_3 \Delta}{a_1} & 2a_3 q_1 q_3 + 2\dfrac{q_0 q_2 \Delta}{a_1} \\ 2a_1 q_1 q_2 + 2\dfrac{q_0 q_3 \Delta}{a_2} & q_0^2 - a_1 q_1^2 + a_2 q_2^2 - a_3 q_3^2 & 2a_3 q_2 q_3 - 2\dfrac{q_0 q_1 \Delta}{a_2} \\ 2a_1 q_1 q_3 - 2\dfrac{q_0 q_2 \Delta}{a_3} & 2a_2 q_2 q_3 + 2\dfrac{q_0 q_1 \Delta}{a_3} & q_0^2 - a_1 q_1^2 - a_2 q_2^2 + a_3 q_3^2 \end{bmatrix} \tag{8}$$

for the ellipsoid $a_1 x^2 + a_2 y^2 + a_3 z^2 = 1$.

Proof. It can be seen that R_θ is a linear transformation and preserves the norm. Using the equalities,

$$\begin{aligned} R_\theta(\mathbf{i}) &= \left(a_1 q_1^2 q_0^2 - q_2^2 a_2 - q_3^2 a_3\right)\mathbf{i} + 2\left(a_1 q_1 q_2 + q_0 q_3\sqrt{\frac{a_1 a_3}{a_2}}\right)\mathbf{j} + 2\left(q_1 q_3 a_1 - q_0 q_2\sqrt{\frac{a_1 a_2}{a_3}}\right)\mathbf{k}, \\ R_\theta(\mathbf{j}) &= 2\left(a_2 q_1 q_2 - q_0 q_3\sqrt{\frac{a_2 a_3}{a_1}}\right)\mathbf{i} + \left(a_2 q_2^2 + q_0^2 - q_1^2 a_1 - q_3^2 a_3\right)\mathbf{j} + 2\left(a_2 q_2 q_3 + q_0 q_1\sqrt{\frac{a_1 a_2}{a_3}}\right)\mathbf{k}, \\ R_\theta(\mathbf{k}) &= 2\left(a_3 q_1 q_3 + q_0 q_2\sqrt{\frac{a_2 a_3}{a_1}}\right)\mathbf{i} + 2\left(a_3 q_2 q_3 - q_0 q_1\sqrt{\frac{a_1 a_3}{a_2}}\right)\mathbf{j} + \left(a_3 q_3^2 + q_0^2 - q_1^2 a_1 - q_2^2 a_2\right)\mathbf{k}, \end{aligned}$$

we can obtain (8). So, the rotation matrix (8) is an elliptical rotation matrix on the ellipsoid $a_1 x^2 + a_2 y^2 + a_3 z^2 = 1$. That is, the equalities $\det R_\theta = 1$ and $R_\theta^t \Omega R_\theta = \Omega$ are satisfied. Also, note that, if we take $a_1 = a_2 = a_3 = 1$, the standard rotation matrix is obtained. Now, let's choose an orthonormal set $\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ satisfying the equalities

$$\mathcal{V}(\varepsilon_0 \times \varepsilon_1) = \varepsilon_2, \quad \mathcal{V}(\varepsilon_2 \times \varepsilon_0) = \varepsilon_1, \quad \mathcal{V}(\varepsilon_1 \times \varepsilon_2) = \varepsilon_0.$$

If ε is a vector in the plane of the ε_0 and ε_1, we can write it as

$$\varepsilon = \cos\alpha\varepsilon_0 + \sin\alpha\varepsilon_1.$$

To compute $R_\theta^q(\boldsymbol{\varepsilon}) = q\boldsymbol{\varepsilon}q^{-1}$, let's find how $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_1$ change under the transformation R_θ^q. Since $\mathbf{V}_q$ is parallel to $\boldsymbol{\varepsilon}_0$, we have $q\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0 q$ by (4) and $R_q(\boldsymbol{\varepsilon}_0) = q\boldsymbol{\varepsilon}_0 q^{-1} = \boldsymbol{\varepsilon}_0 q q^{-1} = \boldsymbol{\varepsilon}_0$. So, $\boldsymbol{\varepsilon}_0$ is not changed under the transformation R_θ^q. It means that $\boldsymbol{\varepsilon}_0$ is the rotation axis. On the other hand,

$$\begin{aligned} R_q(\boldsymbol{\varepsilon}_1) &= q\boldsymbol{\varepsilon}_1 q^{-1} \\ &= (\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta)\,\boldsymbol{\varepsilon}_1 (\cos\theta - \boldsymbol{\varepsilon}_0\sin\theta) \\ &= \boldsymbol{\varepsilon}_1 \cos^2\theta - \cos\theta\sin\theta\,(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0) + \cos\theta\sin\theta\,(\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1) - (\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1)\,\boldsymbol{\varepsilon}_0\sin^2\theta. \end{aligned}$$

Since we know that $\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0 = \mathcal{V}(\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_0) = -\mathcal{V}(\boldsymbol{\varepsilon}_0 \times \boldsymbol{\varepsilon}_1) = -\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1 = -\boldsymbol{\varepsilon}_2$ for orthogonal, pure quaternions, we obtain

$$\begin{aligned} R_q(\boldsymbol{\varepsilon}_1) &= \boldsymbol{\varepsilon}_1\cos^2\theta + (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0)\,\boldsymbol{\varepsilon}_0\sin^2\theta + 2\boldsymbol{\varepsilon}_2\cos\theta\sin\theta \\ &= \boldsymbol{\varepsilon}_1\cos^2\theta + \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0^2\sin^2\theta + 2\boldsymbol{\varepsilon}_2\cos\theta\sin\theta \\ &= \boldsymbol{\varepsilon}_1\cos 2\theta + \boldsymbol{\varepsilon}_2\sin 2\theta \end{aligned}$$

That is, $\boldsymbol{\varepsilon}$ is rotated through the elliptical angle 2θ about $\boldsymbol{\varepsilon}_0$ by the transformation $R_q(\boldsymbol{\varepsilon})$. ■

Corollary 1 *All elliptical rotations on an ellipsoid can be represented by elliptic quaternions which is defined for that ellipsoid.*

The matrix (8) is only rotation matrix in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$ and shows rotations along an ellipse on the $a_1x^2 + a_2y^2 + a_3z^2 = \lambda$, $\lambda \in \mathbb{R}^+$. Here, notice that the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ is unit sphere of the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$. So, R_θ^q rotates a vector elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ or similar to this ellipsoid. Moreover, the matrix R_θ^q depends to elliptical inner product and elliptical vector product and we can write as $R_\theta^q \in \mathbf{SO}_{\mathcal{B}}(n)$ or $R_\theta^q \in \mathbf{SO}_{a_1,a_2,a_3}(n)$ where $q \in \mathbb{H}_{a_1,a_2,a_3}$. We must always use proper elliptic quaternion set and product corresponding to a predetermined ellipsoid.

Example 3 *Let's find the general elliptical rotation matrix for the ellipsoid $2x^2+2y^2+z^2 = 1$. Using (8), we obtain,*

$$R_\theta^q = \begin{bmatrix} q_0^2 + 2q_1^2 - 2q_2^2 - q_3^2 & 4q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 4q_1q_2 & q_0^2 - 2q_1^2 + 2q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 4q_1q_3 - 4q_0q_2 & 4q_0q_1 + 4q_2q_3 & q_0^2 - 2q_1^2 - 2q_2^2 + q_3^2 \end{bmatrix}.$$

Here, $\det R_\theta = \left(q_0^2 + 2q_1^2 + 2q_2^2 + q_3^2\right)^3 = 1$ *and* $R_\theta^t \Omega R = \Omega$ *where* $\Omega = diag\,(2,2,1)$. *For example, the unit quaternion* $q = (0, 1/2, 1/2, 0)$ *represents an elliptical rotation on the ellipsoid* $2x^2+2y^2+z^2 = 1$ *through the elliptical angle* π, *about the axis* $(1/2, 1/2, 0)$. *And the elliptical rotation matrix is*

$$R_\pi^q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Example 4 *Let's find the elliptical rotation matrix which is rotate the point* $A\left(3/2, \sqrt{3}/2, 3/2\right)$ *to* $B\left(-\sqrt{3}/2, -1/2, 3\sqrt{3}/2\right)$ *elliptically on the ellipsoid* $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.
Here, $a_1 = a_2 = 1/4$ *and* $a_3 = 1/9$. *So,* $\Delta = 1/12$. *In example 1, we found the unit rotation axis as* $\boldsymbol{\varepsilon}_0 = \left(1, -\sqrt{3}, 0\right)$ *using the vector product of* $\mathbf{x} = \overrightarrow{OA}$ *and* $\mathbf{y} = \overrightarrow{OB}$ *in* $\mathbb{R}^3_{1/4,1/4,1/9}$.

Also, rotation angle is $\pi/2$ since $\cos\theta = \dfrac{\mathcal{B}(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|_{\mathcal{B}}\|\mathbf{y}\|_{\mathcal{B}}} = 0$. *So, the unit elliptic quaternion representing the elliptical rotation matrix which is rotate A to B is*

$$q = \cos 45^\circ + \boldsymbol{\varepsilon}_0 \sin 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{6}}{2}\mathbf{j}.$$

Therefore, using above Theorem, we find

$$R^q_{\pi/2} = \frac{1}{12}\begin{bmatrix} 3 & -3\sqrt{3} & -4\sqrt{3} \\ -3\sqrt{3} & 9 & -4 \\ 9\sqrt{3} & 9 & 0 \end{bmatrix}.$$

This matrix is same as the rotation matrix obtained using Rodrigues formula in Example 1

Composition of Elliptical Rotations

Let p and q be two unit elliptic quaternions of the same kind. That is, let $p, q \in \mathbb{H}_{a_1,a_2,a_3}$. In this case, $R^p_{\theta_1}$ and $R^q_{\theta_2}$ are two elliptical rotation matrices on the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$. That is $R^p_{\theta_1}$ and $R^q_{\theta_2}$ rotate a vector elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$**.** The composition of these rotations can be expressed by the elliptic quaternion product qp. The axis and the elliptical angle of the composite rotation is given by the product qp. Let $R^p_{\theta_1}(\mathbf{u}) = \mathbf{v}$ and $R^q_{\theta_2}(\mathbf{v}) = \mathbf{w}$. Then

$$\mathbf{w} = R^q_{\theta_2}(\mathbf{v}) = R^q_{\theta_2}\left(R^p_{\theta_1}(\mathbf{u})\right) = qR^p_{\theta_1}(\mathbf{u})\,q^{-1} = qp\mathbf{u}p^{-1}q^{-1} = (qp)\,\mathbf{u}\,(qp)^{-1} = R^{qp}_{\theta_3}(\mathbf{u}).$$

It means that $R^q_{\theta_2}R^p_{\theta_1} = R^{qp}_{\theta_3}$. As an example, let's take the unit elliptic quaternions $q = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ and $p = \left(\frac{1}{6}, \frac{2}{6}, \frac{1}{6}, \frac{5}{6}\right)$. That is, $p, q \in \mathbb{H}_{2,2,1}$. The scalar and the vector products are

$$\mathcal{B}(\mathbf{x},\mathbf{y}) = a_1x_1x_2 + a_2y_1y_2 + a_3z_1z_2,$$
$$\mathcal{V}(\mathbf{x}\times\mathbf{y}) = \left(\frac{\Delta}{a_1}(y_1z_2 - y_2z_1), \frac{\Delta}{a_2}(-x_1z_2 + x_2z_1), \frac{\Delta}{a_3}(x_1y_2 - x_2y_1)\right)$$

respectively. The elliptic quaternion product of the quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as

$$pq = p_0q_0 - \mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) + p_0\mathbf{V}_q + q_0\mathbf{V}_p + \mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q).$$

So, qp can be found to be $qp = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{6}\right) \in \mathbb{H}_{2,2,1}$.
Using (8), we can find as

$$R^q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ R^p = \begin{bmatrix} -1/2 & -1/18 & 11/18 \\ 1/2 & -5/6 & 1/6 \\ 1 & 7/9 & 4/9 \end{bmatrix} \text{ and}$$
$$R^{qp} = \begin{bmatrix} 1/2 & -5/6 & 1/6 \\ -1/2 & -1/18 & 11/18 \\ -1 & -7/9 & -4/9 \end{bmatrix}.$$

It can be seen that the matrix equality $R^qR^p = R^{qp}$ is satisfied.

Remark 1 *If p and q are two unit elliptic quaternions of different kinds, namely $p \in \mathbb{H}_{a_1,a_2,a_3}$ and $q \in \mathbb{H}_{b_1,b_2,b_3}$, then the composition of corresponding rotations cannot be expressed by the elliptic quaternion product. Because, for each quaternion, the scalar product space is different, hence the elliptic quaternion product is also different. $R^p_{\theta_1}$ and $R^q_{\theta_2}$represent a rotations on the ellipsoids $a_1x^2 + a_2y^2 + a_3z^2 = 1$ and $b_1x^2 + b_2y^2 + b_3z^2 = 1$ respectively and the composition of the elliptical rotations on two different ellipsoids cannot be expressed by the elliptical quaternion product of two elliptical quaternions of different kinds.*

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