The roots of a split quaternion

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Abstract

In this paper, we express De Moivre's formula for split quaternions and find roots of a split quaternion using this formula.

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1 Introduction

The roots of a quaternion was given by Niven [8] and Brand [1]. Brand proved De Moivre's thereom and used it to find n-th roots of a quaternion. Using the De Moivre's formula to find roots of a quaternion is more useful way. Euler's and De Moivre's formulas are important for quaternions since every unit quaternion q can be written in the form $q = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ and represents a rotation 2θ about the axis $\boldsymbol{\varepsilon}_0$ in Euclidean 3-space. See [2], [3] information about these formulas for quaternions. In this paper, we express Euler and De Moivre's formulas for split quaternions and examine roots of the a split quaternion with respect to casual character of the split quaternion.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} * \mathbf{j} * \mathbf{k} = 1$. Rotations in Minkowski 3-space can be stated with split quaternions, such as expressing Euclidean rotations by quaternions, . For the detailed information about split quaternions, we refer to references [6], [5], [9], [10]. Split quaternions $\widehat{\mathbb{H}}$ are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Besides, the subspace of $\widehat{\mathbb{H}}$ consisting of pure split quaternions $\widehat{\mathbb{H}}_0$ is identified with the Minkowski 3-space [5]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector product.

2 Preliminaries

It is called that a split quaternion $q = (q_1, q_2, q_3, q_4)$ is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively where $I_q = q_1^2 + q_2^2 - q_3^2 - q_4^2$. The norm of the q is defined as $N_q = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}$. If $N_q = 1$ then q is called unit split quaternion and $q_0 = q/N_q$ is a unit split quaternion for $N_q \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverse and it has the property $q * q^{-1} = q^{-1} * q = 1$. Lightlike quaternions have no inverse. Polar forms of the split quaternions are as follows:

i) Every spacelike quaternion can be written in the form

$$q = N_q \left(\sinh\theta + \boldsymbol{\varepsilon_0} \cosh\theta\right)$$

where $\boldsymbol{\varepsilon}_0 =$ is a spacelike unit vector in $\mathbb{E}_1^3$.

ii) Every timelike quaternion with spacelike vector part can be written in the form

$$q = N_q \left(\cosh\theta + \boldsymbol{\varepsilon_0} \sinh\theta\right)$$

where $\boldsymbol{\varepsilon}_0$ is a spacelike unit vector in $\mathbb{E}_1^3$.

iii) Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q \left(\cos\theta + \boldsymbol{\varepsilon_0} \sin\theta\right)$$

where $\boldsymbol{\varepsilon}_0$ is a timelike unit vector in $\mathbb{E}_1^3$

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Scalar and vector parts of split quaternion q denoted by $Sq = q_1$ and $\mathbf{V}q = q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}$ respectively. Vector parts of the split quaternions identified with the Minkowski 3-space. The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle \mathbf{u}, \mathbf{v} \rangle_L = -u_1v_1 + u_2v_2 + u_1v_2$ where $\mathbf{u} = (u_1, u_1, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\mathbf{u}$ in E_1^3 is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle_L > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_L = 0$ or $\langle \mathbf{u}, \mathbf{u} \rangle_L < 0$ respectively. The norm of the vector $\mathbf{u} \in \mathbb{E}_1^3$ is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle_L|}$.

The split quaternion product of two quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined as

$$p * q = p_1q_1 + \langle \mathbf{V}_p, \mathbf{V}_q \rangle_L + p_1\mathbf{V}_q + q_1\mathbf{V}_p + \mathbf{V}_p \times_L \mathbf{V}_q$$

where $\langle , \rangle_L$ and $\times_L$ are Lorentzian inner product and vector product respectively. Also, we can express a $n-th$ power of a split quaternion $q = S_q + \mathbf{V}_q$ as follows:
i) if n is an even number

$$q^n = \left[\sum_{r=0}^{\frac{n}{2}} \binom{n}{2r} (S)^{n-2r} V^r \right] + \left[\sum_{r=0}^{\frac{n}{2}-1} \binom{n}{2r+1} (S)^{n-2r-1} V^r \right] \mathbf{V}_q$$

ii) if n is an odd number

$$q^n = \left[\sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r} (S)^{n-2r} V^r \right] + \left[\sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r+1} (S)^{n-2r-1} V^r \right] \mathbf{V}_q$$

where $S = S_q$ and $V = \langle \mathbf{V}_q, \mathbf{V}_q \rangle_L$.

The set of timelike quaternions denoted by

$$\mathbb{T}\widehat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_2, q_3, q_4, q_1 \in R,\ I_q > 0\}$$

forms a group under the split quaternion product. Every rotation in the Minkowski 3-space can be expressed with unit timelike quaternions. The set of unit timelike quaternions is represented with $\mathbb{T}\widehat{\mathbb{H}}_1$. If $q = (q_1, q_2, q_3, q_4)$ is a unit timelike quaternion, using the transformation law $(q*\mathbf{V}_r*q^{-1})_i = \sum_{j=1}^{3} R_{ij}(\mathbf{V}_r)_j$ the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_1^2+q_2^2+q_3^2+q_4^2 & 2q_1q_4-2q_2q_3 & -2q_1q_3-2q_2q_4 \\ 2q_2q_3+2q_4q_1 & q_1^2-q_2^2-q_3^2+q_4^2 & -2q_3q_4-2q_2q_1 \\ 2q_2q_4-2q_3q_1 & 2q_2q_1-2q_3q_4 & q_1^2-q_2^2+q_3^2-q_4^2 \end{bmatrix}$$

where $r = (S_r, \mathbf{V}_r)$. These matrices form the 3-dimensional special orthogonal group $\mathbf{SO}(1,2)$. Moreover, the function $\varphi : S_2^3 \simeq \mathbb{T}\widehat{\mathbb{H}}_1 \to \mathbf{SO}(1,2)$ which sends $q = (q_1, q_2, q_3, q_4)$ to matrix R given in (**??**) is an homomorphism of group. The kernel of φ is $\{\pm 1\}$ so that rotation matrix correspond to pairs $\pm q$ of unit quaternion. In particular, $\mathbf{SO}(1,2)$ is isomorphic to the quotient group $\mathbb{T}\widehat{\mathbb{H}}_1 / \{\pm 1\}$ from the first isomorphism theorem. In another words, for every rotation in Minkowski 3-space $\mathbb{E}_1^3$, there are two unit timelike quaternions that determines this rotation. These timelike quaternions are q and $-q$ [10].

3 De Moivre's formula for split quaternions.

In this section we examine De Moivre's formula for split quaternions. For this, we consider causal character of the split quaternion and we specify this formula with respect to timelike and spacelike quaternions separately. In the usual quaternions this formula was given by [2].

3.1 Timelike Quaternions with Spacelike Vector Part ($\mathbb{T}_{Sp}$)

Every timelike quaternion with spacelike vector part can be written in the form

$$q = N_q (\cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta)$$

where $\cosh\theta = \dfrac{|q_1|}{r}$, $\sinh\theta = \dfrac{\sqrt{-q_2^2+q_3^2+q_4^2}}{r}$, $\boldsymbol{\varepsilon}_0 = \dfrac{q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$ and $\boldsymbol{\varepsilon}_0 * \boldsymbol{\varepsilon}_0 = 1$. A unit timelike quaternion q with spacelike vector part represents a rotation of three dimensional nonlightlike Lorentzian vector by an angle hyperbolic angle 2θ about the axis of q [10].

Euler's formula for a unit timelike quaternion with spacelike vector part is hold. Since $\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} = 1$, we have

$$e^{\boldsymbol{\varepsilon}\theta} = \left(1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right) + \boldsymbol{\varepsilon}\left(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right) = \cosh\theta + \boldsymbol{\varepsilon}\sinh\theta.$$

Moreover, it can be show using another method, That is,

$$\begin{aligned} q = \cosh\theta + \boldsymbol{\varepsilon}\sinh\theta &\Rightarrow dq = (\sinh\theta + \boldsymbol{\varepsilon}\cosh\theta)\, d\theta \\ &\Rightarrow dq = \boldsymbol{\varepsilon}(\cosh\theta + \boldsymbol{\varepsilon}\sinh\theta)\, d\theta = \boldsymbol{\varepsilon} q d\theta \end{aligned}$$

Thus, we get $\int \dfrac{dq}{q} = \int \boldsymbol{\varepsilon} d\theta \Rightarrow \ln q = \boldsymbol{\varepsilon}\theta \Rightarrow q = e^{\boldsymbol{\varepsilon}\theta} = \cosh\theta + \boldsymbol{\varepsilon}\sinh\theta$. Now, let's prove De Moivre's formula for a timelike quaternion with spacelike vector part.

Theorem 1 *Let $q = N_q(\cosh\theta + \boldsymbol{\varepsilon}\sinh\theta)$ be a unit timelike quaternion with spacelike vector part. Then,*

$$q^n = (N_q)^n(\cosh(n\theta) + \boldsymbol{\varepsilon}\sinh(n\theta))$$

for $n \in \mathbb{Z}$.

Proof. We use induction on positive integers n. Assume that

$$q^n = (N_q)^n(\cosh(n\theta) + \boldsymbol{\varepsilon}\sinh(n\theta))$$

holds. Then,

$$\begin{aligned} q^{n+1} &= (N_q)^n((\cosh(n\theta)) + \boldsymbol{\varepsilon}\sinh(n\theta))\, N_q(\cosh\theta + \boldsymbol{\varepsilon}\sinh\theta) \\ &= (N_q)^{n+1}(\cosh(n\theta) + \boldsymbol{\varepsilon}\sinh(n\theta))(\cosh\theta + \boldsymbol{\varepsilon}\sinh\theta) \\ &= (N_q)^{n+1}(\cosh(n\theta)\cosh\theta + \sinh(n\theta)\sinh\theta + (\cosh(n\theta)\sinh\theta + \sinh(n\theta)\cosh\theta)\,\boldsymbol{\varepsilon}) \\ &= (N_q)^{n+1}(\cosh((n+1)\theta) + \boldsymbol{\varepsilon}\sinh((n+1)\theta)). \end{aligned}$$

Hence, the formula is true. Moreover, since

$$q^{-1} = (N_q)^{-1}(\cosh\theta - \boldsymbol{\varepsilon}\sinh\theta)$$

and

$$q^{-n} = (N_q)^{-n}(\cosh(n\theta) - \boldsymbol{\varepsilon}\sinh(n\theta)) = (N_q)^{-n}(\cosh(-n\theta) + \boldsymbol{\varepsilon}\sinh(-n\theta)),$$

the formula holds for all integers. ■

3.2 Timelike Quaternions with Timelike Vector Part ($\mathbb{T}_t$)

Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q(\cos\theta + \boldsymbol{\varepsilon}_0\sin\theta)$$

where $\cos\theta = \dfrac{q_1}{r}$, $\sin\theta = \dfrac{\sqrt{q_2^2-q_3^2-q_4^2}}{r}$, $\boldsymbol{\varepsilon}_0 = \dfrac{q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}}{\sqrt{q_2^2-q_3^2-q_4^2}}$ is a timelike unit vector in $\mathbb{E}_1^3$ and $\boldsymbol{\varepsilon}_0 * \boldsymbol{\varepsilon}_0 = -1$. Also, a unit timelike quaternion q with timelike vector part represents a rotation of 3-dimensional nonlightlike Lorentzian vector by an angle 2θ about the axis of q.

Euler's formula for timelike quaternion with timelike vector part is also hold. Since $\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon} * \boldsymbol{\varepsilon} = -1$, we have

$$e^{\boldsymbol{\varepsilon}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + \boldsymbol{\varepsilon}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right) = \cos\theta + \boldsymbol{\varepsilon}\sin\theta.$$

Theorem 2 *Let $q = N_q(\cos\theta + \boldsymbol{\varepsilon}\sin\theta)$ be a unit timelike quaternion with timelike vector part. Then,*

$$q^n = (N_q)^n(\cos n\theta + \boldsymbol{\varepsilon}\sin n\theta)$$

for $n \in \mathbb{N}$.

The proof of this theorem can be done using induction similar to the proof of the Theorem 1.

3.3 Timelike Quaternion with Lightlike Vector Part

Every unit timelike quaternion with null vector part can be written in the form $q = 1 + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ is a null vector. If $q = 1 + \boldsymbol{\varepsilon}$ is a unit timelike quaternion with null vector part, then $q^n = 1 + n\boldsymbol{\varepsilon}$ and only root of the equation $w^n = q$ is $1 + \dfrac{\boldsymbol{\varepsilon}}{n}$.

3.4 Spacelike Quaternions

Every spacelike quaternion can be written in the form

$$q = N_q \left(\sinh\theta + \boldsymbol{\varepsilon}_0 \cosh\theta\right)$$

where $\sinh\theta = \dfrac{q_1}{r}$, $\cosh\theta = \dfrac{\sqrt{-q_2^2 + q_3^2 + q_4^2}}{r}$ and $\boldsymbol{\varepsilon}_0 = \dfrac{q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}}{\sqrt{-q_2^2 + q_3^2 + q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$.

The product of two spacelike quaternions is timelike. That is, for a spacelike quaternion $q = N_q \left(\sinh\theta + \boldsymbol{\varepsilon}_0 \cosh\theta\right)$, $q^2 = N_q^2 \left(\cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta\right)$
Because of that we can give De Moivre's formula for spacelike quaternions as follows.

Theorem 3 *Let $q = N_q \left(\sinh\theta + \boldsymbol{\varepsilon}\cosh\theta\right)$ be a unit spacelike quaternion, then,*

$$q^n = \begin{cases} (N_q)^n \left(\sinh(n\theta) + \boldsymbol{\varepsilon}\cosh(n\theta)\right), & n \text{ is odd} \\ (N_q)^n \left(\cosh(n\theta) + \boldsymbol{\varepsilon}\sinh(n\theta)\right), & n \text{ is even} \end{cases}.$$

4 The roots of a split quaternion

In this section we want to find roots of a split quaternion using the De Moivre's formula given above.

Theorem 4 *Let $q = N_q \left(\cosh\theta + \boldsymbol{\varepsilon}\sinh\theta\right)$ be a timelike quaternion with spacelike vector part, then the equation $w^n = q$ have only one root*

$$w = \sqrt[n]{N_q}\left(\cosh\frac{\theta}{n} + \boldsymbol{\varepsilon}\sinh\frac{\theta}{n}\right)$$

in the set of timelike quaternions $\mathbb{T}\widehat{\mathbb{H}}$.

Proof. If $w^n = q$, q will have the same unit vector as w. So, assume that $w = N\left(\cosh x + \boldsymbol{\varepsilon}\sinh x\right)$ is a root of the equation $w^n = q$. From the Theorem 1, we have

$$w^n = N^n \left(\cosh nx + \boldsymbol{\varepsilon}\sinh nx\right).$$

Thus, $N^n = N_q$ and $x = \dfrac{\theta}{n}$. Therefore, $w = \sqrt[n]{N_q}\left(\cosh\dfrac{\theta}{n} + \boldsymbol{\varepsilon}\sinh\dfrac{\theta}{n}\right)$ is root of the equation $w^n = q$. If we suppose that there are two roots satisfying the equality, we obtain that these roots must be equal to each other. ■

Let's find the roots of the equation $w^3 = q$ where $q = \sqrt{2} + \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{k}$. Since $I_q = 1$, q is a unit timelike quaternion and it's vector part is a spacelike vector. Then, we can write as $q = \cosh\theta + \varepsilon\sinh\theta$. Indeed, $q = \sqrt{2} + \left(\sqrt{2}, \sqrt{2}, -1\right)1 \Rightarrow \cosh\theta = \sqrt{2}$ and $\sinh\theta = 1$.
From the last equalities, we find $\theta = \ln\left(\sqrt{2} + 1\right)$. Thus,

$$q = \cosh\ln\left(\sqrt{2} + 1\right) + \boldsymbol{\varepsilon}\sinh\ln\left(\sqrt{2} + 1\right)$$

where $\boldsymbol{\varepsilon} = \left(\sqrt{2}, \sqrt{2}, -1\right)$. And using above Lemma, we find the root as

$$w = \left(\cosh\frac{\ln\left(\sqrt{2} + 1\right)}{3} + \boldsymbol{\varepsilon}\sinh\frac{\ln\left(\sqrt{2} + 1\right)}{3}\right).$$

Theorem 5 *Let $q = N_q \left(\cos\theta + \boldsymbol{\varepsilon}\sin\theta\right)$ be a timelike quaternion with timelike vector part, then the equation $w^n = q$ have n roots in the timelike quaternions $\mathbb{T}\widehat{\mathbb{H}}$. and they are*

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{\theta + 2m\pi}{n} + \boldsymbol{\varepsilon}\sin\frac{\theta + 2m\pi}{n}\right)$$

where $m = 0, 1, 2, ..., n-1$.

Proof. We assume that $w = N(\cos\varphi + \boldsymbol{\varepsilon}\sin\varphi)$ is a root of the equation, since vector parts of w and q are same, So, we find

$$N_q = N^n, \cos n\varphi = \cos\theta and \sin n\varphi = \sin\theta$$

using Theorem 2. Thus, nth roots of q are

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{\theta+2m\pi}{n} + \boldsymbol{\varepsilon}\sin\frac{\theta+2m\pi}{n}\right)$$

for $m = 0, 1, 2, ..., n-1$. ■

Let's find the roots of the equation $w^3 = -1 + \sqrt{3}\mathbf{i}$. $q = -1 + \sqrt{3}\mathbf{i}$ is a timelike quaternion with timelike part such that $N_q = 2$.Then, q can be written as

$$q = 2\left(\cos\left(\frac{2\pi}{3} + 2m\pi\right) + \boldsymbol{\varepsilon}\sin\left(\frac{2\pi}{3} + 2m\pi\right)\right)$$

for $m \in N^+$. From the theorem 5, the roots of the equation $w^3 = q$ are

$$w_m = \sqrt[3]{2}\left(\cos\frac{\frac{2\pi}{3}+2m\pi}{3} + \boldsymbol{\varepsilon}\sin\frac{\frac{2\pi}{3}+2m\pi}{3}\right)$$

for $m = 0, 1, 2$. So,

$$w_0 = \sqrt[3]{2}\left(\cos\frac{2\pi}{9} + \boldsymbol{\varepsilon}\sin\frac{2\pi}{9}\right),$$
$$w_1 = \sqrt[3]{2}\left(\cos\frac{8\pi}{9} + \boldsymbol{\varepsilon}\sin\frac{8\pi}{9}\right),$$
$$w_3 = \sqrt[3]{2}\left(\cos\frac{14\pi}{9} + \boldsymbol{\varepsilon}\sin\frac{14\pi}{9}\right)$$

are the roots of the given equation.

Now, we examine the solution of the equations in the form $w^n = q$ such that $q \in \mathbb{R}$. We consider $q \in \mathbb{R}$ regard as a timelike quaternion with timelike vector part.

Theorem 6 *Let $q \in \mathbb{R}$ be a unit timelike quaternion with $\boldsymbol{\varepsilon} = \vec{0}$ then there are infinitely many roots of the equation $w^n = q$ in the set of timelike quaternions $\mathbb{T}\widehat{\mathbb{H}}$.*

Proof. We must choose w such that *n-th* power of it must be a real number with zero vector part. Since we consider $q \in \mathbb{R}$ regard as a timelike quaternion with timelike vector part, we can write w as $w = N(\cos\theta + \boldsymbol{\varepsilon}\sin\theta)$, so $w^n = N^n(\cos n\theta + \boldsymbol{\varepsilon}\sin n\theta) = q$ and $N = \sqrt[n]{N_q}$, $\theta = \dfrac{2m\pi}{n}$ for $m = 0, 1, 2$. Thus, we find roots as

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{2m\pi}{n} + \boldsymbol{\varepsilon}\sin\frac{2m\pi}{n}\right), \quad \text{for } m = 0, 1, 2, ..., n-1.$$

Here, as if there are n roots of the equation actually there are infinite roots. Because, every quaternion in the form

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{2m\pi}{n} + \boldsymbol{\varepsilon}\sin\frac{2m\pi}{n}\right), \quad \text{for } m = 0, 1, 2, ...$$

with a timelike unit vector $\boldsymbol{\varepsilon}$ is a root of the equation and we choose infinite $\boldsymbol{\varepsilon}$. That is, for every timelike vector $\boldsymbol{\varepsilon}$, we find another timelike quaternion which is satisfying the equation. ■

Remark 7 *All of the timelike quaternions satisfying the equation $w^n = q$ represent a rotation through same angle about different axes in the Minkowski 3-space.*

For example, the roots of the equation $w^4 = -4$ are

$$w_m = \sqrt[4]{4}\left(\cos\frac{\pi+2k\pi}{4} + \boldsymbol{\varepsilon}\sin\frac{\pi+2k\pi}{4}\right)$$

for $k = 0, 1, 2, 3$. But, for every timelike vector $\boldsymbol{\varepsilon}$, the equality $(w_m)^n = -4$ holds. That is, we can choose infinitely timelike quaternion satisfying the equation providing that its norm is 4 and vector part is any timelike vector. $w \in [(1,3,2,2), (1,1,0,0), (1,2,1,\sqrt{2}), ... \}$

Here, each of the quaternions $(1,3,2,2), (1,1,0,0), (1,2,1,\sqrt{2}), ...$ represents a rotation through 90 degree about the different axis $\boldsymbol{\varepsilon} \in \{(3,2,2), (1,0,0), (2,1,\sqrt{2}), ...\}$.

Remark 8 *If n is an even number, then some of the roots of the equation $w^n = q \in \mathbb{R}^+$ can be a spacelike quaternion. For example, a spacelike quaternion $w \in \{(0,0,1,0),(0,0,0,1),(0,2,2,1),...\}$ is a root of the $w^4 = 1$. But, observe that we solve the equation $w^n = q$ in the set of timelike quaternions in thereom 6.*

Now let's state what the roots of a spacelike quaternion are.in the set of spacelike quaternions. For this case, it is important that n is odd or even, according to theorem 3.

Theorem 9 *Let $q = N_q(\sinh\theta + \boldsymbol{\varepsilon}\cosh\theta)$ be a spacelike quaternion, then solution of the equation $w^n = q$ in the spacelike quaternions*
i) doesn't exist if n is an even number
ii) have only one spacelike quaternion root $w = \sqrt[n]{N_q}\left(\sinh\frac{\theta}{n} + \boldsymbol{\varepsilon}\cosh\frac{\theta}{n}\right)$ if n is an odd number.

Proof. If n is even number, *n-th* power of a spacelike quaternion will be a timelike quaternion and in this case there is no solution.
So, let $w = N(\sinh\varphi + \boldsymbol{\varepsilon}\cosh\varphi)$ be a root of the equation $w^n = q$ such that n is an odd number, then

$$w^n = N^n(\sinh n\varphi + \boldsymbol{\varepsilon}\cosh n\varphi) = N_q(\sinh\theta + \boldsymbol{\varepsilon}\cosh\theta)$$

and we have $\varphi = \frac{\theta}{n}$ and $N = \sqrt[n]{N_q}$. ■

Let's find the roots of the equation $x^3 = 1 + \sqrt{2}\mathbf{k}$. Since $I_q = -1$, q is a unit spacelike quaternion. Then, it can be written in the form $q = \sinh\theta + \boldsymbol{\varepsilon}\cosh\theta$ where $\boldsymbol{\varepsilon} = (0,0,1)$, $\cosh\theta = \sqrt{2}$ and $\sinh\theta = 1$. So,

$$x = \sinh\frac{\ln\left(\sqrt{2}+1\right)}{3} + \boldsymbol{\varepsilon}\cosh\frac{\ln\left(\sqrt{2}+1\right)}{3}$$

is the only root of the equation $x^3 = 1 + \sqrt{2}k$.

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