A generalization of the array type polynomials

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Abstract

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We introduce a generalization of the array type polynomials by using two specific generating functions and investigate some of its basic properties in the sequel. A recurrence relation and two summation formulas involving these polynomials are also given.

Keywords: Appell polynomials, array polynomials, Stirling numbers of the second kind, Apostol–Bernoulli polynomials, generating function, recurrence relation.

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1 Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\,e^{xt} = \sum_{n=0}^{\infty} A_n(x)\,\frac{t^n}{n!},$$

where $f(t)$ is a formal power series in t, have found many considerable applications in mathematics, theoretical physics and chemistry [1,20]. One the well-known examples of these polynomials is the Bernoulli polynomials for $f(t) = \dfrac{t}{e^t - 1}$.

Another example known in the literature as the generalized Apostol–Bernoulli polynomials can be defined as [8,9,10]

$$\left(\frac{t}{\lambda e^t - 1}\right)^m e^{xt} = \sum_{n=0}^{\infty} B_n^{(m)}(x,\lambda)\frac{t^n}{n!}, \tag{1.1}$$

where

$$(\lambda \in \mathbb{C};\ \ |t| < 2\pi \text{ if } \lambda = 1;\ \ |t| < |\log \lambda| \ \text{ if } \lambda \neq 1).$$

For $x = 0$ in (1.1), the generalized Apostol–Bernoulli numbers is derived. See also [12,13,18,19] in this sense.

On the other hand, the generalized $\lambda-$Stirling numbers of the second kind $S_m^n(\lambda)$ were introduced in [11] as

$$\frac{(\lambda e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_m^n(\lambda)\,\frac{t^n}{n!}, \quad \text{for } \lambda \in \mathbb{C} \text{ and } m \in \mathbb{N}_0 = \{0,1,2,...\}, \tag{1.2}$$

where $\lambda = 1$ gives the well known Stirling numbers of the second kind.

By referring to (1.2), the $\lambda-$array type polynomials $S_m^n(x,\lambda)$ are defined as [3]

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}=\sum_{n=0}^{\infty}S_m^n(x,\lambda)\frac{t^n}{n!}, \tag{1.3}$$

see also [4,16,17]. In [2], the authors deduced some notable identities associated with $\lambda-$array type polynomials, $\lambda-$Stirling numbers of the second kind and the Apostol–Bernoulli numbers. Furthermore, they studied $\lambda-$array polynomials via $\lambda-$delta operator. For more details see [5, 6, 7] and references therein.

In this paper, we define two parametric kinds of $\lambda-$array type polynomials as follows:

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}\cos yt=\sum_{n=0}^{\infty}S_m^{n,c}\left(x,y,\lambda\right)\frac{t^n}{n!}, \tag{1.4}$$

and

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}\sin yt=\sum_{n=0}^{\infty}S_m^{n,s}\left(x,y,\lambda\right)\frac{t^n}{n!}.$$

Motivated by some of above-mentioned papers, we then introduce a bivariate kind of $\lambda-$array type polynomials and investigate its general properties. We also derive a recurrence relation and some summation formulas including these polynomials and other special polynomials and numbers such as parametric-kind Apostol–Bernoulli, generalized Apostol–Bernoulli numbers, and generalized $\lambda-$Stirling numbers of the second kind.

2 A family of $\lambda-$array type polynomials

For two sequences $\{a_n\}_n$ and $\{b_n\}_n$ such that

$$A(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\quad\text{and}\quad B(t)=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},$$

their binomial convolution is given by

$$c_n=a_n*b_n=\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}.$$

Also, we have the following generating function for the sequence $\{c_n\}_n$ as

$$C(t) = A(t)B(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

By noting these comments, the Taylor–Maclaurin expansions of the two functions $e^{xt}\cos yt$ and $e^{xt}\sin yt$ are explicitly computed as (see e.g. [14] and [15])

$$e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{2.1}$$

and

$$e^{xt}\sin(yt) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!},$$

where $C_n(x,y)$ and $S_n(x,y)$ are defined respectively as follows:

$$C_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k},$$

and

$$S_n(x,y) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1},$$

in which $[x]$ stands for the greatest integer of x.

By observing the definitions of $\lambda-$Stirling numbers of the second kind $S_m^n(\lambda)$, $C_n(x,y)$ and $S_n(x,y)$, we can now introduce two parametric kinds of $\lambda-$array type polynomials as follows

$$S_m^{n,c}(x,y,\lambda) = S_m^n(\lambda) * C_n(x,y),$$

and

$$S_m^{n,s}(x,y,\lambda) = S_m^n(\lambda) * S_n(x,y),$$

whose generating functions are respectively defined by

$$\sum_{n=0}^{\infty} S_m^{n,c}(x,y,\lambda)\frac{t^n}{n!} = \frac{(\lambda e^t - 1)^m}{m!} e^{xt}\cos yt,$$

and

$$\sum_{n=0}^{\infty} S_m^{n,s}(x,y,\lambda)\frac{t^n}{n!} = \frac{(\lambda e^t-1)^m}{m!}e^{xt}\sin yt,$$

from which it is readily deduced that

$$S_m^{n,c}(x,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k} S_m^k(\lambda)\, C_{n-k}(x,y),$$

and

$$S_m^{n,s}(x,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k} S_m^k(\lambda)\, S_{n-k}(x,y).$$

Theorem 2.1 *For every $n \in \mathbb{N}$, we have*

$$S_m^{n,c}(2x,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k} S_m^{k,c}(x,y,\lambda)\, x^{n-k}, \tag{2.2}$$

and

$$S_m^{n,s}(2x,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k} S_m^{k,s}(x,y,\lambda)\, x^{n-k}. \tag{2.3}$$

Proof. From (1.4), one can write

$$\begin{aligned}\sum_{n=0}^{\infty} S_m^{n,c}(2x,y,\lambda)\frac{t^n}{n!} &= \frac{(\lambda e^t-1)^m}{m!}e^{2xt}\cos yt \\ &= \left(\sum_{n=0}^{\infty} S_m^{n,c}(x,y,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} S_m^{k,c}(x,y,\lambda)\, x^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients $\frac{t^n}{n!}$ of both sides yields the desired identity (2.2). Equality (2.3) can be obtained similarly. ■

Theorem 2.2 *For $n \in \mathbb{N}$ and $z \in \mathbb{R}$, we have*

$$S_m^{n,c}(x+z,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k} S_m^{k,c}(x,y,\lambda)\, z^{n-k}$$

and

$$S_m^{n,s}(x+z,y,\lambda)=\sum_{k=0}^{n}\binom{n}{k}S_m^{k,s}(x,y,\lambda)\,z^{n-k}.$$

Proof. Using (1.4), one has

$$\begin{aligned}\sum_{n=0}^{\infty}S_m^{n,c}(x+z,y,\lambda)\frac{t^n}{n!}&=\frac{(\lambda e^t-1)^m}{m!}e^{(x+z)t}\cos yt\\&\left(\sum_{n=0}^{\infty}S_m^{n,c}(x,y,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}S_m^{k,c}(x,y,\lambda)\,z^{n-k}\right)\frac{t^n}{n!},\end{aligned}$$

which gives the first identity after equating the coefficients $\frac{t^n}{n!}$. The second one can be derived in a similar way. ■

Theorem 2.3 *For $r,n\in\mathbb{N}$ with $n\geq r$, the following derivative formulas hold:*

$$\frac{\partial^r}{\partial x^r}\left(S_m^{n,c}(x,y,\lambda)\right)=\langle n\rangle_r S_m^{n-r,c}(x,y,\lambda),\tag{2.4}$$

$$\frac{\partial^r}{\partial x^r}\left(S_m^{n,s}(x,y,\lambda)\right)=\langle n\rangle_r S_m^{n-r,s}(x,y,\lambda),\tag{2.5}$$

where $\langle x\rangle_r$ denotes the falling factorial, defined for $x\in\mathbb{R}$ by

$$\langle x\rangle_r=\prod_{k=0}^{r-1}(x-k)=\begin{cases}x(x-1)...(x-r+1), & \text{if } r\geq 1;\\ 1, & \text{if } r=0.\end{cases}$$

Proof. Using (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial^r}{\partial x^r}\left(S_m^{n,c}(x,y,\lambda)\right)\frac{t^n}{n!}&=\frac{\partial^r}{\partial x^r}\frac{(\lambda e^t-1)^m}{m!}e^{xt}\cos yt\\&=\sum_{n=0}^{\infty}\langle n\rangle_r S_m^{n-r,c}(x,y,\lambda)\frac{t^n}{n!},\end{aligned}$$

which concludes the proof of (2.4). The assertion (2.5) can be proved similarly. ■

Corollary 2.4 *For $n \in \mathbb{N}$, the following relations are valid:*

$$\frac{\partial}{\partial y} S_m^{n,c}(x,y,\lambda) = -nS_m^{n-1,s}(x,y,\lambda),$$

and

$$\frac{\partial}{\partial y} S_m^{n,s}(x,y,\lambda) = nS_m^{n-1,c}(x,y,\lambda).$$

Theorem 2.5 *For $n \in \mathbb{N}$, two parametric kinds of $\lambda-$array type polynomials $S_m^{n,c}(x,y,\lambda)$ and $S_m^{n,s}(x,y,\lambda)$ satisfy the following recurrent relations:*

$$\lambda S_m^{n,c}(x+1,y,\lambda) = (m+1)S_{m+1}^{n,c}(x,y,\lambda) + S_m^{n,c}(x,y,\lambda), \tag{2.6}$$

and

$$\lambda S_m^{n,s}(x+1,y,\lambda) = (m+1)S_{m+1}^{n,s}(x,y,\lambda) + S_m^{n,s}(x,y,\lambda). \tag{2.7}$$

Proof. From (1.4), we have

$$\begin{aligned}
&\lambda \sum_{n=0}^{\infty} S_m^{n,c}(x+1,y,\lambda)\frac{t^n}{n!} \\
&= \lambda \frac{(\lambda e^t - 1)^m}{m!} e^{(x+1)t}\cos yt \\
&= \frac{(\lambda e^t - 1)^m}{m!} e^{xt}\cos yt\left\{\lambda e^t - 1 + 1\right\} \\
&= (m+1)\frac{(\lambda e^t - 1)^{m+1}}{(m+1)!} e^{xt}\cos yt + \frac{(\lambda e^t - 1)^m}{m!} e^{xt}\cos yt \\
&= (m+1)\sum_{n=0}^{\infty} S_{m+1}^{n,c}(x,y,\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} S_m^{n,c}(x,y,\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients $\frac{t^n}{n!}$ on both sides of the last equality leads to desired identity (2.6). The relation (2.7) follows easily in a similar way. ■

Theorem 2.6 *The following summation formulas holds true:*

$$\sum_{k=0}^{n+m} \binom{n+m}{k} S_m^{n+m-k,c}(x,y,\lambda) B_k^{(m)}(\lambda) = \begin{cases} \binom{n+m}{m} C_n(x,y), & \text{if } n \geq 0; \\ 0, & \text{if } -m \leq n \leq -1, \end{cases} \tag{2.8}$$

and

$$\sum_{k=0}^{n+m}\binom{n+m}{k}S_m^{n+m-k,s}\left(x,y,\lambda\right)B_k^{(m)}\left(\lambda\right)=\begin{cases}\binom{n+m}{m}S_n\left(x,y\right), & \text{if } n\geq 0;\\ 0, & \text{if } -m\leq n\leq -1,\end{cases} \tag{2.9}$$

where $B_k^{(m)}\left(\lambda\right)$ *is the generalized Apostol–Bernoulli numbers.*

Proof. Consider the equality

$$\left(\frac{t}{\lambda e^t-1}\right)^m\frac{\left(\lambda e^t-1\right)^m}{m!}e^{xt}\cos yt=\frac{t^m}{m!}e^{xt}\cos yt.$$

Making use of (1.1) for $x=0$, (1.4) and (2.1), we find that

$$\sum_{n=0}^{\infty}B_n^{(m)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_m^{n,c}\left(x,y,\lambda\right)\frac{t^n}{n!}=\frac{t^m}{m!}\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!}.$$

So, we have

$$\sum_{n=-m}^{\infty}\sum_{k=0}^{n+m}\binom{n+m}{k}B_k^{(m)}(\lambda)S_m^{n+m-k,c}\left(x,y,\lambda\right)\frac{t^n}{(n+m)!}=\frac{1}{m!}\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!} \tag{2.10}$$

Now, if we compare the coefficients t^n on both sides of (2.10), then, we reach the formula (2.8). The relation (2.9) can be derived similarly. ■

Theorem 2.7 *The following relations are valid*

$$\sum_{k=0}^{n+m}\binom{n+m}{k}B_{k,c}\left(x,y,\lambda\right)S_m^{n+m-k}\left(\lambda\right)=\binom{n+m}{m}S_{m-1}^{n+m-1,c}\left(x,y,\lambda\right),$$

and

$$\sum_{k=0}^{n+m}\binom{n+m}{k}B_{k,s}\left(x,y,\lambda\right)S_m^{n+m-k}\left(\lambda\right)=\binom{n+m}{m}S_{m-1}^{n+m-1,s}\left(x,y,\lambda\right),$$

where $B_{k,c}\left(x,y,\lambda\right)$ *and* $B_{k,s}\left(x,y,\lambda\right)$ *are two parametric kinds of Apostol–Bernoulli polynomials, presented in [19, Eq.8], and* $S_m^n\left(\lambda\right)$ *is the generalized* $\lambda-$*Stirling numbers of the second kind, given by (1.2).*

Proof. Consider the equality

$$\frac{1}{t^m}\left(\frac{t}{\lambda e^t-1}\right)\left(\lambda e^t-1\right)^m e^{xt}\cos yt=\left(\frac{\lambda e^t-1}{t}\right)^{m-1} e^{xt}\cos yt.$$

Now, utilizing [19, Eq.8], (1.2) and (1.4) yields that

$$\frac{1}{t^m}\left(\sum_{n=0}^{\infty}B_{n,c}(x,y,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}m!S_m^n\left(\lambda\right)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}(m-1)!S_{m-1}^{n,c}\left(x,y,\lambda\right)\frac{t^{n-m+1}}{n!}. \tag{2.11}$$

Thus, while the left hand side of (2.11) can be written as

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_{k,c}(x,y,\lambda)}{k!}\frac{m!S_m^{n-k}\left(\lambda\right)}{(n-k)!}\right)t^{n-m}\\&=\sum_{n=-m}^{\infty}\left(\sum_{k=0}^{n+m}\frac{B_{k,c}(x,y,\lambda)}{k!}\frac{m!S_m^{n+m-k}\left(\lambda\right)}{(n+m-k)!}\right)t^n,\end{aligned} \tag{2.12}$$

the right hand side can be expressed as

$$\sum_{n=-(m-1)}^{\infty}(m-1)!\frac{S_{m-1}^{n+m-1,c}\left(x,y,\lambda\right)}{(n+m-1)!}t^n. \tag{2.13}$$

Note that for $n=-m$, since $S_m^0\left(\lambda\right)B_{0,c}(x,y,\lambda)=0$, comparing the coefficients t^n of (2.12) and (2.13) gives the first identity of the theorem. The second one can be obtained in a similar way. ■

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