SOME GEOMETRIC APPLICATIONS OF TIMELIKE QUATERNIONS

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Abstract

In this paper, we introduce some properties of split quaternions and show that every great hyperbolic arc on the H_0^2 corresponds to a timelike quaternion with spacelike vector part. Also, we prove the sine and cosine laws for hyperboloidal triangles on the H_0^2 using this property. Aim of this paper is to express some geometric applications of timelike quaternions in the hyperbolic geometry similar to geometric applications of quaternions in the spherical geometry.

Keywords : Lorentzian Geometry, Quaternions, Split quaternions, The sine and cosine laws, Hyperbolic Trigonometry.

MSC 2000: 53B30, 15A66.

1 Introduction

Quaternion algebra $\mathbb{H}$ is an associative, non-commutative division ring with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{i} * \mathbf{j} = \mathbf{k}$, $\mathbf{j} * \mathbf{k} = \mathbf{i}$, $\mathbf{k} * \mathbf{i} = \mathbf{j}$, $\mathbf{j} * \mathbf{i} = -\mathbf{k}$, $\mathbf{k} * \mathbf{j} = -\mathbf{i}$, $\mathbf{i} * \mathbf{k} = -\mathbf{j}$ [7]. We write any quaternion in the form $q = (q_1, q_2, q_3, q_4) = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ or $q = S_q + \mathbf{V}_q$ where the symbols $S_q = q_1$ and $\mathbf{V}_q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ denote the scalar and vector parts of q. The conjugate of the quaternion q is denoted by K_q, and defined as $K_q = S_q - \mathbf{V}_q$. The norm of a quaternion $q = (q_1, q_2, q_3, q_4)$ is defined by $\sqrt{q * K_q} = \sqrt{K_q * q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$ and denoted by N_q and we say that $q_0 = q/N_q$ is a unit quaternion where $q \neq 0$. Every unit quaternion can be written in the form $q_0 = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ where $\boldsymbol{\varepsilon}_0$ is a unit vector satisfying the equality $\boldsymbol{\varepsilon}_0^2 = \boldsymbol{\varepsilon}_0 * \boldsymbol{\varepsilon}_0 = -1$ and is called axis of the quaternion [7].

Each great circle arc AB of a sphere centered at O, where $OA = \mathbf{a}$ and $OB = \mathbf{b}$ are two unit vectors perpendicular to $\boldsymbol{\varepsilon}_0$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ corresponds to a unit quaternion. That is, q_0 corresponds to an arc of great circle whose plane is normal to $\boldsymbol{\varepsilon}_0$ and whose central angle θ has the positive sense relative $\boldsymbol{\varepsilon}_0$. Using this property of quaternions, some spherical trigonometric relations as sine and cosine laws for spherical triangles can be proved [1].

In this paper, firstly, we give basic notions of the Lorentzian space. Also, we introduce the concept of split quaternion and give the definitions of the timelike, spacelike and lightlike split quaternions in section 3. In the last section, we show that each great hyperbolic arc on the unit hyperboloid H_0^2 and S_1^2 corresponds to a timelike quaternion with spacelike vector part and each great ellipse arc on the S_1^2 corresponds to a timelike quaternion with timelike vector part. At last, using these properties we prove the sine and cosine laws for hyperboloidal triangles on the H_0^2. Here, a great hyperbola, ellipse or circle implies with a trace on the surface of the H_0^2 or S_1^2 of a plane that passes through the center of the H_0^2 or S_1^2.

2 Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\mathbf{u} = (u_1, u_1, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a Lorentzian vector $\mathbf{u}$ in $\mathbb{E}_1^3$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle_L > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_L = 0$ or $\langle \mathbf{u}, \mathbf{u} \rangle_L < 0$ respectively. The norm of the vector $\mathbf{u} \in \mathbb{E}_1^3$ is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle_L|}$. Also, for the timelike vectors in the Minkowski 3-space, we say that a timelike vector is future pointing or past pointing if the first component of the vector is positive or negative respectively, the Lorentzian vector product $\mathbf{u} \times_L \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined as follows:

$$\mathbf{u} \times_L \mathbf{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Moreover, for the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ in the Minkowski 3-space, the equalities

$$\mathbf{x} \times_L (\mathbf{y} \times_L \mathbf{z}) = \langle \mathbf{x}, \mathbf{y} \rangle_L \mathbf{z} - \langle \mathbf{x}, \mathbf{z} \rangle_L \mathbf{y}$$

$$\langle \mathbf{x} \times_L \mathbf{y}, \mathbf{z} \times_L \mathbf{w} \rangle_L = - \begin{vmatrix} \langle \mathbf{x}, \mathbf{z} \rangle_L & \langle \mathbf{x}, \mathbf{w} \rangle_L \\ \langle \mathbf{y}, \mathbf{z} \rangle_L & \langle \mathbf{y}, \mathbf{w} \rangle_L \end{vmatrix}$$

are satisfied. Proof of these identities can be done using vector analysis. The hyperbolic and Lorentzian unit spheres are

$$H_0^2 = \left\{ \mathbf{a} \in \mathbb{E}_1^3 : \langle \mathbf{a}, \mathbf{a} \rangle_L = -1 \right\} \text{ and } S_1^2 = \left\{ \mathbf{a} \in \mathbb{E}_1^3 : \langle \mathbf{a}, \mathbf{a} \rangle_L = 1 \right\}$$

respectively. There are two components of H_0^2 passing through $(1, 0, 0)$ and $(-1, 0, 0)$ a future pointing hyperbolic sphere and a past pointing hyperbolic unit sphere, and they are denoted by H_0^{2+} and H_0^{2-} respectively.

Theorem 1 *Let u and $\mathbf{v}$ be vectors in the Minkowski 3-space.*
i) If $\mathbf{u}$ and $\mathbf{v}$ are future pointing (or past pointing) timelike vectors, then $\mathbf{u} \times_L \mathbf{v}$ is a spacelike vector, $\langle \mathbf{u}, \mathbf{v} \rangle_L = -\|\mathbf{u}\| \|\mathbf{v}\| \cosh\theta$ and $\|\mathbf{u} \times_L \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sinh\theta$ where θ is the hyperbolic angle between $\mathbf{u}$ and $\mathbf{v}$.
ii) If $\mathbf{u}$ and $\mathbf{v}$ are spacelike vectors satisfying the inequality $|\langle \mathbf{u}, \mathbf{v} \rangle_L| < \|\mathbf{u}\| \|\mathbf{v}\|$ then, $\mathbf{u} \times_L \mathbf{v}$ is timelike, $\langle \mathbf{u}, \mathbf{v} \rangle_L = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$ and $\|\mathbf{u} \times_L \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
iii) If $\mathbf{u}$ and $\mathbf{v}$ are spacelike vectors satisfying the inequality $|\langle \mathbf{u}, \mathbf{v} \rangle_L| > \|\mathbf{u}\| \|\mathbf{v}\|$ then, $\mathbf{u} \times_L \mathbf{v}$ is spacelike, $\langle \mathbf{u}, \mathbf{v} \rangle_L = \|\mathbf{u}\| \|\mathbf{v}\| \cosh\theta$ and $\|\mathbf{u} \times_L \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sinh\theta$ where θ is the hyperbolic angle between $\mathbf{u}$ and $\mathbf{v}$.
iv) If $\mathbf{u}$ and $\mathbf{v}$ are spacelike vectors satisfying the equality $|\langle \mathbf{u}, \mathbf{v} \rangle_L| = \|\mathbf{u}\| \|\mathbf{v}\|$ then, $\mathbf{u} \times_L \mathbf{v}$ is lightlike.

For further Lorentzian concepts see [8],[4] and [2].

3 Split Quaternions

The semi-Euclidean 4-space with 2-index represented with $\mathbb{E}_2^4$. The inner product of this semi-Euclidean space is

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{E}_2^4} = -u_1 v_1 - u_2 v_2 + u_3 v_3 + u_4 v_4$$

and we say that $\mathbf{u}$ is timelike, spacelike or lightlike if $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{E}_2^4} < 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{E}_2^4} > 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{E}_2^4} = 0$ for the vector $\mathbf{u}$ in $\mathbb{E}_2^4$ respectively. Split quaternions $\widehat{\mathbb{H}}$ are identified with the semi-Euclidean

space $\mathbb{E}_2^4$. Besides, the subspace of $\widehat{\mathbb{H}}$ consisting of pure split quaternions $\widehat{\mathbb{H}}_0$ is identified with the Minkowski 3-space [5]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector products.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \mathbf{k}^2 = 1$ and $\mathbf{i} * \mathbf{j} = \mathbf{k}$, $\mathbf{j} * \mathbf{k} = -\mathbf{i}$, $\mathbf{k} * \mathbf{i} = \mathbf{j}$, $\mathbf{j} * \mathbf{i} = -\mathbf{k}$, $\mathbf{k} * \mathbf{j} = \mathbf{i}$, $\mathbf{i} * \mathbf{k} = -\mathbf{j}$. The split quaternion algebra is the even subalgebra of the Clifford algebra of 3-dimensional Lorentzian space. Scalar and vector parts of split quaternion q denoted by $S_q = q_1$ and $\mathbf{V}_q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ respectively. The split quaternion product of two quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined as

$$p * q = p_1 q_1 + \langle \mathbf{V}p, \mathbf{V}_q \rangle_L + p_1 \mathbf{V}_q + q_1 \mathbf{V}p + \mathbf{V}p \times_L \mathbf{V}_q$$

where $\langle , \rangle_L$ and $\times_L$ are Lorentzian inner product and vector product respectively [10]. If $S_q = 0$ then, q is called pure split quaternion. Split quaternion product of two pure split quaternions $p = (p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k})$ and $q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ is

$$p * q = \langle \mathbf{V}p, \mathbf{V}_q \rangle_L + \mathbf{V}p \times_L \mathbf{V}_q = -p_2q_2 + p_3q_3 + p_4q_4 + \begin{vmatrix} -i & j & k \\ p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \end{vmatrix}. \qquad (*)$$

Let $q = (q_1, q_2, q_3, q_4) = S_q + \mathbf{V}_q$ be a split quaternion. The conjugate of a split quaternion, denoted K_q, is defined as $K_q = S_q - \mathbf{V}_q$. The conjugate of the sum of quaternions is the sum of their conjugates. Since the vector parts of q and K_q differ only in sign, we have $I_q \overset{def}{=} q * K_q = K_q * q$. Also, for pure split quaternions, since changing the sign of determinant in $(*)$ is equivalent to interchanging the second and third rows, $K(\mathbf{V}_q * \mathbf{V}_{q'}) = \mathbf{V}_{q'} * \mathbf{V}_q$. We say that a split quaternion q is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively where $I_q = q * K_q = K_q * q$. Obviously, $-I_q = -q_1^2 - q_2^2 + q_3^2 + q_4^2$ is identified with $\langle q, q \rangle_{\mathbb{E}_2^4}$ for the split quaternion $q = (q_1, q_2, q_3, q_4)$. The norm of the $q = (q_1, q_2, q_3, q_4)$ is defined as $N_q = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}$. If $N_q = 1$ then q is called unit split quaternion and $q_0 = q/N_q$ is a unit split quaternion for $N_q \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverses and they hold the property $q * q^{-1} = q^{-1} * q = 1$. And they are constructed as $q^{-1} = \dfrac{K_q}{I_q}$. Lightlike quaternions have no inverses [10].

The set of timelike quaternions denoted by

$$\mathbb{T}\widehat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_2, q_3, q_4, q_1 \in R,\ I_q > 0\}$$

forms a group under the split quaternion product. A similar relation to the relationship between unit quaternions and rotations in the Euclidean space exists between unit timelike quaternions and rotations in the Minkowski 3-space. Every rotation in the Minkowski 3-space can be expressed with unit timelike quaternions [10].

Vector part of any spacelike quaternion is spacelike since $q_1^2 + q_2^2 - q_3^2 - q_4^2 < 0$ and $0 < q_1^2 < -q_2^2 + q_3^2 + q_4^2 =< \mathbf{V}_q, \mathbf{V}_q >_L$. But, vector part of any timelike quaternion can be spacelike, timelike and null. Because of that we examine timelike quaternions whether the vector part is spacelike, timelike or null in $\mathbb{E}_1^3$. This is important especially for polar forms and rotations.

i) Every spacelike quaternion can be written in the form

$$q = N_q \left(\sinh\theta + \boldsymbol{\varepsilon}_0 \cosh\theta\right)$$

where $\sinh\theta = \frac{q_1}{N_q}$, $\cosh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{N_q}$ and $\boldsymbol{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$.

ii) Every timelike quaternion with spacelike vector part can be written in the form

$$q = N_q \left(\cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta\right)$$

where $\cosh\theta = \frac{|q_1|}{N_q}$, $\sinh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{N_q}$, $\boldsymbol{\varepsilon}_0 = \frac{q_2 i+q_3 j+q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$ and $\boldsymbol{\varepsilon}_0^2 = 1$.

For example, for the timelike quaternion $q = (2,1,0,2)$, the polar form is $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta = 2 + \frac{(1,0,2)}{\sqrt{3}}\sqrt{3}$.

iii) Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q \left(\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta\right)$$

where $\cos\theta = \frac{q_1}{N_q}$, $\sin\theta = \frac{\sqrt{q_2^2-q_3^2-q_4^2}}{N_q}$, $\boldsymbol{\varepsilon}_0 = \frac{q_2 i+q_3 j+q_4 k}{\sqrt{q_2^2-q_3^2-q_4^2}}$ is a timelike unit vector in $\mathbb{E}_1^3$ and $\boldsymbol{\varepsilon}_0^2 = -1$.

For example, for the timelike quaternion $q = (1,2,1,1)$, the polar form is $q = \sqrt{3}\left(\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta\right) = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{(2,1,1)}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{3}}.\right)$.

4 Geometry of Timelike Quaternions

Each great circle arc of a unit sphere in the Euclidean 3-space corresponds to a unit quaternion. All such arcs of this great circle are equally valid representation. In this section, we examine the relation between unit timelike quaternions and the Lorentzian unit spheres H_0^2 and S_1^2 and we show that each great hyperbolic arc on the unit hyperboloid H_0^2 and S_1^2 corresponds to a timelike quaternion with spacelike vector part and each great ellipse arc on the S_1^2 corresponds to a timelike quaternion with timelike vector part. Using these properties we prove the sine and cosine laws for hyperboloidal triangles on the H_0^2. Here, a great hyperbola, ellipse or circle implies with a trace on the surface of the H_0^2 or S_1^2 of a plane that passes through the center of the H_0^2 or S_1^2
Considering that a vector in the Lorentzian space are a split quaternion with scalar part is zero, we express following theorems.

Theorem 2 *Every unit timelike quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ with spacelike vector part can be expressed in the form $\mathbf{v} * \mathbf{u}^{-1}$ such that θ is the hyperbolic angle between Lorentzian vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying one of the following conditions.*
i) $\mathbf{u}$ and $\mathbf{v}$ are unit timelike vectors which are perpendicular to spacelike unit vector $\boldsymbol{\varepsilon}_0$
ii) $\mathbf{u}$ and $\mathbf{v}$ are unit spacelike vectors which satisfy the inequality $|\langle \mathbf{u}, \mathbf{v}\rangle_L| > 1$ and perpendicular to spacelike unit vector $\boldsymbol{\varepsilon}_0$.

Proof. Let's take the timelike quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ with spacelike vector part.

$$\begin{aligned} q &= \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta \\ &= |\mathbf{u}|\,|\mathbf{v}| \cosh\theta + \boldsymbol{\varepsilon}_0 |\mathbf{u}|\,|\mathbf{v}| \sinh\theta \end{aligned}$$

Considering to the Theorem 1,

$$\begin{aligned} q &= -\langle \mathbf{u}, \mathbf{v}\rangle_L + (\mathbf{u} \times_L \mathbf{v}) \\ &= -S(\mathbf{u} * \mathbf{v}) + \mathbf{V}(\mathbf{u} * \mathbf{v}) \\ &= -S(\mathbf{v} * \mathbf{u}) - \mathbf{V}(\mathbf{v} * \mathbf{u}) \\ &= -\mathbf{v} * \mathbf{u} = \mathbf{v} * K\mathbf{u} = \mathbf{v} * \mathbf{u}^{-1}. \quad \blacksquare \end{aligned}$$

For example, the unit timelike quaternion $q = (3,-8,-6,-6)$ with spacelike vector part can be expressed as $\mathbf{v} * \mathbf{u}^{-1}$ such that $\mathbf{u} = (9,8,4)$ and $\mathbf{v} = (3,2,2)$ are unit future pointing timelike

vectors satisfying the equalities $\cosh\theta = -\langle \mathbf{u}, \mathbf{v}\rangle_L = 3$, $\sinh\theta = \sqrt{8}$ and $\boldsymbol{\varepsilon}_0 = \dfrac{\mathbf{u}\times_L \mathbf{v}}{\|\mathbf{u}\times_L \mathbf{v}\|} = \dfrac{1}{\sqrt{8}}(-8,-6,-6)$.

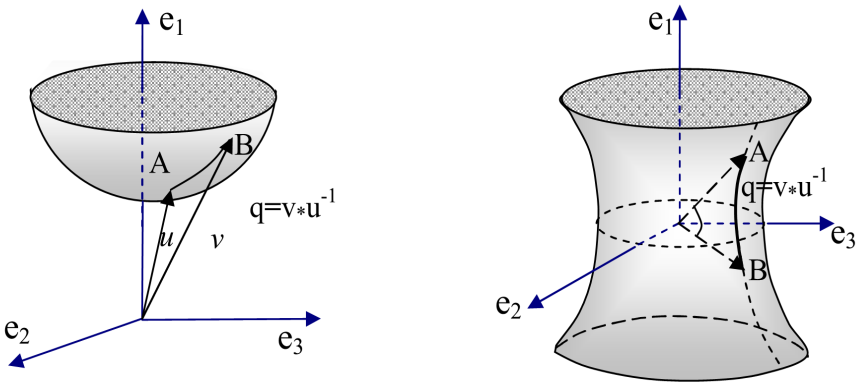
Also, for the unit timelike quaternion $q = (-9,0,-4,8)$ with spacelike vector part can be expressed as $\mathbf{v} * \mathbf{u}^{-1}$ such that $\mathbf{u} = (2,2,1)$ and $\mathbf{v} = (-2,2,1)$ are unit spacelike vectors satisfying the inequality $|\langle \mathbf{u}, \mathbf{v}\rangle_L| > 1$ and the equalities $\boldsymbol{\varepsilon}_0 = \dfrac{\mathbf{u}\times_L \mathbf{v}}{\|\mathbf{u}\times_L \mathbf{v}\|} = \dfrac{\sqrt{5}}{20}(0,-4,8)$: and $\cosh\theta = \langle \mathbf{u}, \mathbf{v}\rangle_L = 9$.
We can express the following corollaries from the Theorem 2.

Corollary 3 *A great hyperbolic arc of the H_0^{2+} constructed with the unit future pointing timelike vectors $\mathbf{u}, \mathbf{v}$ corresponds to a unit timelike quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ with spacelike vector part and where θ is the hyperbolic angle between $\mathbf{u}$ and $\mathbf{v}$. Moreover, the corresponding split quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ is the product $\mathbf{v} * \mathbf{u}^{-1}$.*

For example, for the great hyperbolic arc constructed with the unit future pointing timelike vectors $\mathbf{u} = (3,2,2)$ and $\mathbf{v} = (8,7,4)$, corresponding unit timelike quaternion is

$$ArcAB \cong q = \mathbf{v} * \mathbf{u}^{-1} = (2,6,4,5).$$

Also, $ArcBA \cong q^{-1} = \mathbf{u} * \mathbf{v}^{-1} = (2,-6,-4,-5)$.



Each great hyperbolic arc on the H_0^2 or S_1^2 corresponds to a timelike quaternion with spacelike vector part.

Corollary 4 *A great hyperbolic arc of the S_1^2 constructed with unit spacelike vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying the inequality $|\langle \mathbf{u}, \mathbf{v}\rangle| > 1$ corresponds to a unit timelike quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ with spacelike vector part where θ is the hyperbolic angle between $\mathbf{u}$ and $\mathbf{v}$. Moreover, the corresponding split quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ is $\mathbf{v} * \mathbf{u}^{-1}$.*

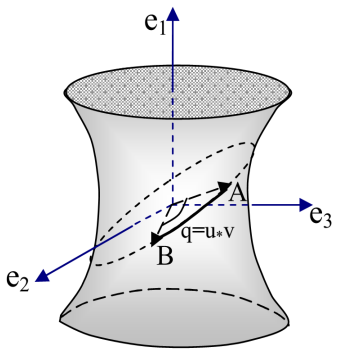
For example, for the great hyperbolic arc constructed with the unit spacelike vectors $\mathbf{u} = (2,1,2)$ and $\mathbf{v} = (1,1,1)$, corresponding unit timelike quaternion is $ArcAB \cong q = \mathbf{v} * \mathbf{u}^{-1} = (-1,-1,0,1)$. Also, $ArcBA \cong q^{-1} = \mathbf{u} * \mathbf{v}^{-1} = (-1,1,0,-1)$.

If the timelike unit quaternion $q = \cosh\theta + \boldsymbol{\varepsilon}_0 \sinh\theta$ corresponds to the arc AB, $q^{-1} = \cosh\theta - \boldsymbol{\varepsilon}_0 \sinh\theta$ corresponds to an arc having the same plane and hyperbolic angle, but whose sense is positive relative to $-\boldsymbol{\varepsilon}_0$. Hence q^{-1} corresponds to the arc BA.

Now, let's take into account a unit timelike quaternion $q = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ with timelike vector part. For the unit spacelike vectors $\mathbf{u}$ and $\mathbf{v}$, we can write the unit quaternion $q = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ as $q = |\mathbf{u}|\,|\mathbf{v}|\cos\theta + \varepsilon_0 |\mathbf{u}|\,|\mathbf{v}|\sin\theta$. If we consider the Theorem 1, we can see $q =< \mathbf{u}, \mathbf{v} >_L + \mathbf{u}\times_L \mathbf{v}$. It means that q is the split quaternion product of $\mathbf{u}$ and $\mathbf{v}$.Thus, we can state following theorem.

Theorem 5 *Every unit timelike quaternion* $q = \cos\theta + \varepsilon_0 \sin\theta$ *with timelike vector part can be expressed in the form* $\mathbf{u} * \mathbf{v}$ *where* $\mathbf{u}$ *and* $\mathbf{v}$ *are unit spacelike vectors which are perpendicular to a timelike unit vector* ε_0 *and* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

For example, the unit timelike quaternion $q = (0, -3, -2, -2)$ with timelike vector part can be expressed as $\mathbf{u} * \mathbf{v}$ such that $\mathbf{u} = (2, 2, 1)$ and $\mathbf{v} = (2, 1, 2)$ are unit spacelike vectors satisfying the inequality $|\langle \mathbf{u}, \mathbf{v} \rangle_L| < 1$ and the equalities $\boldsymbol{\varepsilon} = (-3, -2, -2) = \mathbf{u} \times_{\mathbf{L}} \mathbf{v}$ and $\cos\theta = \langle \mathbf{u}, \mathbf{v} \rangle_L = 0$.



Each great ellipse arc on the S_1^2 corresponds to a timelike quaternion with timelike vector part.

Corollary 6 *An arc of the great ellipse of the* S_1^2 *constructed with unit spacelike vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *satisfying the inequality* $|\langle \mathbf{u}, \mathbf{v} \rangle_L| < 1$ *corresponds to a unit timelike split quaternion* $q = \cos\theta + \varepsilon_0 \sin\theta$ *with timelike vector part where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$. *Moreover,the corresponding split quaternion* $q = \cos\theta + \varepsilon_0 \sin\theta$ *is* $\mathbf{u} * \mathbf{v}$.

If the unit timelike quaternion $q = \cos\theta + \varepsilon_0 \sin\theta$ corresponds to the ellipse arc AB, $q^{-1} = \cos\theta - \varepsilon_0 \sin\theta$ corresponds to same ellipse arc having the same plane and angle, but whose sense is positive relative to $-\varepsilon_0$. Hence q^{-1} corresponds to the arc BA.

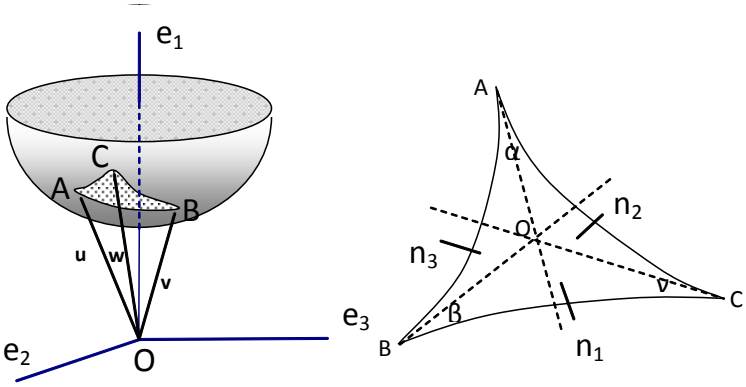
The sine and cosine laws for the spherical triangle on the unit sphere constructed with great circle arcs having a, b, c lenghts are

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma} \quad \text{and} \quad \cos c = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

where α, β, γ are angles between these great circle arcs. These equalities can be easily proved using the vector analysis and spherical trigonometry. Besides, the proof of these laws can be given as an application of the the fact that a quaternion corresponds to an arc of the great circle of the unit sphere.

4.1 A New Proof for Hyperbolic Sine and Cosine Formulas

Now, the fact that using the each great hyperboloidal arc of the unit hyperboloid H_0^{2+} corresponds to a timelike quaternion with spacelike vector part, we will give the proof of the sine and cosine laws for hyperboloidal triangles on the H_0^{2+}. Proof of this laws also can be found in [9] and [6]. Also, for the sine and cosine law in the Lorentzian plane, see [2] and [3].



Hyperboloidal Triangle

Assume an arbitrary hyperboloidal triangle on the H_0^{2+} with vertices at A, B and C with opposite sides a, b, c and let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ denote unit future pointing timelike vectors directed from the center of the hyperboloid to the vertices A, B, C respectively. Also, let α, β and γ be angles between tangent vectors of the sides (hyperbolic arcs) at the vertices A, B and C of hyperbolic triangle. That is, $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OB}$ and $\mathbf{w} = \overrightarrow{OC}$ are unit future pointing timelike vectors on H_0^{2+}, Theorem 3 shows that the hyperbolic arcs AB, BC and AC can be represented by the split quaternions $\mathbf{v} * \mathbf{u}^{-1}$, $\mathbf{w} * \mathbf{v}^{-1}$ and $\mathbf{w} * \mathbf{u}^{-1}$ respectively. Let's write $p = \mathbf{v} * \mathbf{u}^{-1}$, $q = \mathbf{w} * \mathbf{v}^{-1}$ then,

$$\mathbf{w} * \mathbf{u}^{-1} = \mathbf{w} * \mathbf{v}^{-1} * \mathbf{v} * \mathbf{u}^{-1} = q * p$$

and the equation $\widetilde{AB} + \widetilde{BC} = \widetilde{AC}$ may be written $arc(p) + arc(q) = arc(q * p)$. Moreover, in general, the vector sum of any number of great hyperbolic arcs on the H_0^{2+} or H_0^{2-} is given by the arc corresponding to the product of their representative timelike quaternions taken in reverse order. Using these notations, let's prove the sine and cosine laws for hyperboloidal triangles.

Let's consider the hyperboloidal triangle ABC on H_0^{2+} where $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OB}$ and $\mathbf{w} = \overrightarrow{OC}$. Then,

$$\widetilde{BC} \cong \mathbf{w} * \mathbf{v}^{-1} = \cosh a + \mathbf{n}_1 \sinh a$$
$$\widetilde{CA} \cong \mathbf{u} * \mathbf{w}^{-1} = \cosh b + \mathbf{n}_2 \sinh b$$
$$\widetilde{AB} \cong \mathbf{v} * \mathbf{u}^{-1} = \cosh c + \mathbf{n}_3 \sinh c$$

where $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$ are spacelike vectors, a, b, c are hyperboloidal angles between $\mathbf{w}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{u}$ respectively. The vector equation

$$\widetilde{CA} + \widetilde{AB} = \widetilde{CB}$$

corresponds to the quaternion equation

$$\left(\mathbf{v} * \mathbf{u}^{-1}\right) * \left(\mathbf{u} * \mathbf{w}^{-1}\right) = \mathbf{v} * \mathbf{w}^{-1}.$$

Let's write the corresponding timelike quaternions.

$$\begin{aligned}(\cosh c + \mathbf{n}_3 \sinh c) * (\cosh b + \mathbf{n}_2 \sinh b) &= \cosh c \cosh b + \\ &+ \mathbf{n}_2 \cosh c \sinh b + \mathbf{n}_3 \cosh b \sinh c + \\ &+ (\mathbf{n}_3 * \mathbf{n}_2) \sinh b \sinh c \\ &= \cosh a - \mathbf{n}_1 \sinh a.\end{aligned}$$

Since $\mathbf{n}_3 * \mathbf{n}_2 = \langle \mathbf{n}_3, \mathbf{n}_2 \rangle_L + \mathbf{n}_3 \times_L \mathbf{n}_2 = \cos\alpha - \mathbf{u} \sin\alpha$ such that α is the angle between the spacelike vectors $\mathbf{n}_2$ and $\mathbf{n}_3$, we have

$$\cosh c\cosh b+\mathbf{n}_2\cosh c\sinh b+\mathbf{n}_3\cosh b\sinh c+\sinh b\sinh c\cos\alpha-\mathbf{u}\sin\alpha\sinh b\sinh c=\cosh a-\mathbf{n}_1\sinh a$$

and equating scalar parts of both sides, we have

$$\cosh c\cosh b+\sinh b\sinh c\cos\alpha=\cosh a$$

Thus, we find

$$\cos\alpha=\frac{\cosh a-\cosh c\cosh b}{\sinh b\sinh c}.$$

This is cosine law for hyperbolic triangle.
Also, equating the vector parts of both sides, we have

$$\mathbf{n}_2\cosh c\sinh b+\mathbf{n}_3\cosh b\sinh c-\mathbf{u}\sin\alpha\sinh b\sinh c=-\mathbf{n}_1\sinh a.$$

Taking Lorentzian inner product of both sides by the timelike vector $\mathbf{u}$, and since $\mathbf{u}$ is perpendicular to both of the $\mathbf{n}_2$ and $\mathbf{n}_3$, we obtain

$$\sin\alpha\sinh b\sinh c=-\left\langle\mathbf{u},\mathbf{n}_1\sinh a\right\rangle_L=-\left\langle\mathbf{u},\mathbf{v}\wedge\mathbf{w}\right\rangle_L=-\left(\mathbf{u},\mathbf{v},\mathbf{w}\right)$$

Thus,

$$\frac{\sin\alpha}{\sinh a}=\frac{-\left(\mathbf{u},\mathbf{v},\mathbf{w}\right)}{\sinh a\sinh b\sinh c}.$$

Since the right member is unchanged by a cyclical permutation, we obtain

$$\frac{\sin\alpha}{\sinh a}=\frac{\sin\beta}{\sinh b}=\frac{\sin\gamma}{\sinh c}.$$

Thus, we proved sine and cosine laws for hyperbolic triangles H_0^{2+} using timelike quaternions with spacelike vector part. □

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