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ORIGINAL PAPER

Integral representations of Apostol-type splines: approach to generating function method of special polynomials

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Abstract
The purpose of this paper is to give moment formulas with the aid of Milovanović [16]. Other aims are to establish new integral formulas in order to define new Apostol-type splines in terms of the Apostol-type polynomials. By the aid of these integral formulas, we derive a novel class of moment-type expressions arising from integrals of these polynomials. By applying generating function techniques and moment computations, we derive explicit representations and approximation formulas for Apostol- Bernoulli, Euler, and Frobenius spline polynomials. Closed-form expansions are established using Goldman's formula and symbolic moment identities. The connection between cardinal B-splines $\phi_n(x)$ and uniform B-splines $N_{0,n-1}(x)$ is given. We compute integrals using beta-type representations and provide recurrence relations for numerical implementation. Furthermore, we develop a comparative numerical table that confirms the validity of the approximation.

Keywords Apostol-type polynomials · Catalan numbers · Stirling numbers · Beta and gamma functions · Special functions · Spline · Generating functions · B-spline moments · Integral representations

Mathematics Subject Classification 05A15 · 11B83 · 65D07 · 11S80 · 11B68 · 05A19 · 05A10 · 26C10

1 Introduction

In this paper, we define new families of Apostol spline-based polynomials obtained via integrals involving B-splines and Apostol-type polynomials. We give novel integral representations and generating functions. Moreover, we offer unifying generalizations for Apostol-type Bernoulli, Apostol-type Euler, and Apostol-type Frobenius-Euler structures, providing both theoretical insight and computational advantages. These results yield new integral identities, recurrence relations, and algebraic expressions that establish connections between spline theory. In this work, we construct new families of Apostol-type polynomials by integrating B-splines and cardinal B-splines into their definitions. Specifically, we define novel Apostol-type spline Bernoulli, Euler, and Frobenius polynomials using convolution

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integrals of the form

$$\mathfrak{F}_n^{(\alpha)}(x;\lambda) = \int_0^1 N_{0,n}(t) F_n^{(\alpha)}(x-t;\lambda)\, dt,$$

where $N_{0,n}(t)$ denotes the uniform B-spline of order n, and $F_n^{(\alpha)}$ denote the corresponding Apostol-type polynomial. By involving symbolic expansions and moment-based representations, we derive new analytical identities and computational tables for evaluating these spline polynomials. In particular, we exploit the structure of cardinal B-splines $\phi_n(x)$ as orthogonal weight functions, following the approach of Milovanović, and apply moment Gaussian quadrature to achieve evaluations. The theoretical approach is complemented by recurrence relations derived from Schoenberg-type identities and convolution integrals involving polynomials. B-spline functions have been given as basis functions for constructing smooth curves, including computer-aided geometric design and finite element methods. Among their numerous applications, they play a vital role in numerical approximation, computer-aided geometric design, and the analysis of special functions. Spline functions also play an important role in numerical analysis, approximation theory, and mathematical modeling. In recent years, there has been growing interest in connecting spline theory with special functions, number theory, and combinatorial identities. The main objective of this study is to unify the theory of Apostol-type polynomials and B-spline basis functions with the integral. Inspired by the generating function constructions of Goldman [6], we introduce a novel family of Apostol-type spline polynomials, constructed using uniform B-splines. The combination of these two structures yields polynomial families that retain both the smoothness and local support of splines and the algebraic structure of the Apostol family. Also, Apostol-type polynomials such as the Apostol-Bernoulli, Apostol-Euler, and Apostol-Frobenius polynomials form a special functions with connections to number theory, combinatorics, and mathematics. These polynomials exhibit recurrence relations, summation identities, and integral representations. Additionally, we evaluate these expressions numerically using a Gaussian tailored for cardinal B-spline weight functions. The quadrature weights are constructed using the orthogonalization methods described by Milovanović [16], allowing for accurate approximation of spline-based integrals.

This research not only enhances the analytical properties of spline functions but also provides new insights into the structure of special number sequences such as Apostol Bernoulli, Euler, and Frobenius polynomials. This work yields a family of spline-based polynomials defined via Apostol-type polynomials. These results yield new integral identities, recurrence relations, and algebraic expressions that establish connections between spline theory.

The rising factorial (Pochhammer symbol) is given by

$$(\alpha)^{(n)} = \begin{cases} \alpha\,(\alpha+1)\ldots(\alpha+n-1), & n \in \mathbb{N} \\ 1, & n = 0, \end{cases}$$

where $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, (*cf.* [1]-[32]).

The falling factorial is given by

$$(\alpha)_{(n)} = \begin{cases} \alpha\,(\alpha-1)\ldots(\alpha-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

(*cf.* [1]-[32]).

The Frobenius–Euler numbers and polynomials of higher order are defined by, respectively,

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{1}$$

and

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{2}$$

where $\lambda\neq 1$(*cf.* [4, 23]).

In Ref. [9], Gun and Simsek defined the following monic polynomials $u_n^{(\alpha)}(x;\lambda)$ with real variable x:

$$\left(\frac{1+\lambda}{\lambda e^t+1}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}u_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{3}$$

where $|t|<\pi$ when $\lambda=1$; $|t|<|\log(-\lambda)|$ when $\lambda\neq 1$, $1^{\alpha}=1$ and also $\lambda\neq -1$.

$$u_n^{(\alpha)}(x;\lambda)=H_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right) \tag{4}$$

and

$$\mathfrak{q}_n(\lambda;\alpha)=(1+\lambda)^n u_n^{(\alpha)}(\lambda) \tag{5}$$

(*cf.* [9]).

The Apostol-Bernoulli numbers and polynomials of higher order are defined, respectively, as follows:

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{6}$$

and

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{7}$$

where $|t+\ln(\lambda)|<2\pi;\Re(\alpha)>0$ when $\lambda\neq 1$ (*cf.* [2, 23, 31]).

The Apostol-Euler numbers and polynomials of higher order are defined, respectively, as follows:

$$\frac{2^{\alpha}}{(\lambda e^t+1)^{\alpha}}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{8}$$

and

$$\frac{2^{\alpha}}{(\lambda e^t+1)^{\alpha}}e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{9}$$

where $|2t+\ln(\lambda)|<\pi;\Re(\alpha)<0$ when $\lambda\neq -1$ (*cf.* [2, 9, 31]). Putting $\alpha\to-\alpha$ into Eq. (8), we have the Apostol-Euler numbers of higher negative order

$$\frac{(\lambda e^t+1)^{\alpha}}{2^{\alpha}}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(-\alpha)}(\lambda)\frac{t^n}{n!}, \tag{10}$$

where $|2t+\ln(\lambda)|<\pi;\Re(\alpha)>0$ when $\lambda\neq -1$ (*cf.* [2, 9]).

The beta-type rational functions $\mathfrak{M}_{j,n}(x)$ are defined by means of the following generating function:

$$\left(\frac{x}{x+1}\right)^{j} e^{t(x+1)} = \sum_{n=0}^{\infty} \mathfrak{M}_{j,n}(x)\frac{t^n}{n!}$$

(*cf.* [24]).

The beta-type rational functions are also defined by the following formula:

$$\mathfrak{M}_{j,n}(\lambda) = \lambda^{j}\,(\lambda+1)^{n-j}\,, \tag{11}$$

where $n,\ j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\lambda \in \mathbb{R}$ denote set of real numbers, (or $\mathbb{C}$ denote set of complex numbers)

$$\int_{-1}^{0} \mathfrak{M}_{j,n}(\lambda)d\lambda = (-1)^{j}\,\frac{1}{(n+1)\binom{n}{j}} \tag{12}$$

(*cf.* [24]).

The λ-Stirling numbers of the second kind which are defined by means of the following generating function:

$$\frac{\left(\lambda e^{t}-1\right)^{v}}{v!} = \sum_{n=0}^{\infty} S(n, v; \lambda)\frac{t^n}{n!} \tag{13}$$

(*cf.* [23, 31]).

Substituting $\lambda = 1$ into Eq. (13), we have the Stirling numbers of the second kind

$$S(n, v; 1) = S_2(n, v) \tag{14}$$

and

$$x^{n} = \sum_{k=0}^{n} S_2(n, k)(x)_{(k)}$$

(*cf.* [14, 23, 31]).

The Stirling numbers of the first kind, $S_1\,(n, k)$ are defined by

$$(\alpha)_{(n)} = \sum_{k=0}^{n} S_1\,(n, k)\,\alpha^{k} \tag{15}$$

(*cf.* [14, 23, 31]).

In Ref. [25], Simsek defined the following generating function for the combinatorial numbers, $y_1(n, k; \lambda)$:

$$\frac{\left(\lambda e^{t}+1\right)^{v}}{v!} = \sum_{n=0}^{\infty} y_1(n, k; \lambda)\frac{t^n}{n!}, \tag{16}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ and, also

$$k^{n}\lambda^{k} = \sum_{l=0}^{k} (-1)^{k-l}\binom{k}{l} l!\, y_1(n, l; \lambda) \tag{17}$$

(*cf.* [25]). This numbers are also known as the Simsek numbers ([11, 18]).

In Ref. [9], Gun and Simsek showed that the polynomials $\mathfrak{q}_n(\lambda;\alpha)$ are also related to the beta-type rational functions and the Stirling numbers of the second kind:

$$\mathfrak{q}_n(\lambda;\alpha)=\sum_{j=1}^{n}(-1)^j S_2(n,j)(\alpha)^{(n)}\mathfrak{M}_{j,n}(\lambda). \tag{18}$$

The Catalan numbers C_n are defined by

$$C_n=\frac{1}{n+1}\binom{2n}{n} \tag{19}$$

(*cf.* [12, 32]).

Simsek [28] defined generating functions for the following infinite sums of power binomial coefficients:

$$B_v(n,\lambda,p)=\sum_{j=0}^{\infty}\frac{j^v\lambda^j}{\binom{n}{j}^p} \tag{20}$$

(*cf.* [28]).

Goldman [6] constructed the following generating function for the uniform B-splines $N_{0,n}(x;p)$:

$$G_0(x,t;p)=\sum_{j=0}^{p}(-1)^j\left[\frac{(x-j)^j t^j}{j!}+\frac{(x-j)^{j-1}t^{j-1}}{(j-1)!}\right]e^{(x-j)t}=\sum_{n=0}^{\infty}N_{0,n}(x;p)t^n, \tag{21}$$

where $p\le x\le p+1$.

Recently, with the aid of the Bernstein basis function, Simsek [29] gave many applications of the generating equation for the uniform B-splines $N_{0,n}(x;p)$, given by (21). In the literature, the uniform B-splines $N_{0,n}(x;p)$ are also represented by various notations (Refs. [5, 6, 22]). Using Eq. (21), Goldman [6] proved the following well-known identity, known as Schoenberg's identity:

$$N_{0,n}(x;p)=\frac{1}{n!}\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}(x-j)^n, \tag{22}$$

which holds for $p\le x\le p+1$, and the de Boor recurrence relation:

$$N_{0,n}(x;p)=\frac{x}{n}N_{0,n-1}(x;p)+\frac{n+1-x}{n}N_{1,n-1}(x;p) \tag{23}$$

(*cf.* [5, 20, 21]). For $x\le 1$, we have $N_{1,n}(x)=0$. Therefore,

$$N_{0,n}(x)=\frac{x}{n}N_{0,n-1}(x),$$

where $0\le x\le 1$.

Thus, we get

$$N_{0,0}(x)=1,\quad N_{0,1}(x)=x,\quad N_{0,2}(x)=\frac{x^2}{2!},\ldots,N_{0,n}(x)=\frac{x^n}{n!} \tag{24}$$

(*cf.* [5, 20, 21]).

2 Formulas and identities of integrals from special polynomials

In this section, using methods from integral equations and finite sums, we present some formulas and identities involving the polynomials $\mathfrak{q}_n(\lambda;\alpha)$ which are related to Beta-type rational functions and the Stirling numbers of the second kind.

Theorem 1 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\lambda = \sum_{j=1}^{n} S_2(n,j)(\alpha)^{(n)} \frac{1}{(n+1)\binom{n}{j}}. \tag{25}$$

Proof By integrating both sides of Eq. (18) with respect to λ from -1 to 0, we obtain

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\lambda = \sum_{j=1}^{n} (-1)^j S_2(n,j)(\alpha)^{(n)} \int_{-1}^{0} \mathfrak{M}_{j,n}(\lambda) d\lambda.$$

Combining the above equation with Eq. (12), after performing several calculations, the desired result is achieved. □

By setting $\alpha = \frac{1}{2}$ in Eq. (25), using the following formula:

$$\left(\frac{1}{2}\right)^{(n)} = \frac{(n+1)!}{4^n} C_n, \tag{26}$$

we have the following corollary:

Corollary 1 *Let $n \in \mathbb{N}_0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n\left(\lambda;\frac{1}{2}\right) d\lambda = \sum_{j=1}^{n} S_2(n,j) \frac{n! C_n}{4^n \binom{n}{j}}.$$

Substituting $\alpha = 1$ into Eq. (25), we also get the following corollary:

Corollary 2 *Let $n \in \mathbb{N}_0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;1)\, d\lambda = -\sum_{j=1}^{n} \frac{S_2(n,j) n!}{(n+1)\binom{n}{j}}.$$

By integrating both sides of Eq. (18) with respect to λ from -1 to 0, and using Eq. (11), we obtain

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\lambda = \sum_{j=1}^{n} (-1)^j S_2(n,j)(\alpha)^{(n)} \int_{-1}^{0} \lambda^j (1+\lambda)^{n-j}\, d\lambda.$$

After some necessary calculations in the above equation, we derive the following theorem:

Theorem 2 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\lambda = \sum_{j=1}^{n} S_2(n,j)(\alpha)^{(n)} \beta(j+1, n-j+1),$$

where $\beta\left(z_1,z_2\right)$ *is the Beta function:*

$$\beta\left(z_1,z_2\right)=\int_0^1 t^{z_1-1}\left(1-t\right)^{z_2-1}dt.$$

By combining Theorem 2 with the following well-known relation

$$\beta\left(z_1,z_2\right)=\frac{\Gamma\left(z_1\right)\Gamma\left(z_2\right)}{\Gamma\left(z_1+z_2\right)},$$

where $\Gamma\left(z\right)$ is the gamma function:

$$\Gamma\left(n\right)=\left(n-1\right)!,$$

for positive integer n, we arrive at Eq. (25).

Corollary 3 *Let* $n\in\mathbb{N}_0$ *and* $\alpha\in\mathbb{C}$ *with* $\Re(\alpha)>0$. *Then, we have*

$$\int_{-1}^{0}\mathfrak{q}_n\left(\lambda;\alpha\right)d\lambda=\sum_{j=1}^{n}S_2(n,\,j)(\alpha)^{(n)}\frac{\Gamma\left(j+1\right)\Gamma\left(n-j+1\right)}{\Gamma\left(n+2\right)}. \tag{27}$$

Theorem 3 *Let* $n\in\mathbb{N}_0$. *Then, we have*

$$B_0\left(n,-1,\,1\right)=\begin{cases}\frac{2(n+1)}{n+2}, & n\ even\\ 0, & n\ odd.\end{cases}$$

Proof By summing both sides of Eq. (12) in terms of λ, we get

$$\sum_{j=0}^{n}\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda=\frac{1}{(n+1)}\sum_{j=0}^{n}\frac{(-1)^j}{\binom{n}{j}}.$$

Combining the above equation with Eq. (20), we have

$$\sum_{j=0}^{n}\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda=\frac{1}{(n+1)}B_0\left(n,-1,\,1\right). \tag{28}$$

Applying left hand side of the above equation, we have

$$\int_{-1}^{0}\sum_{j=0}^{n}\mathfrak{M}_{j,n}(\lambda)d\lambda=\int_{-1}^{0}\sum_{j=0}^{n}\lambda^j\left(\lambda+1\right)^{n-j}d\lambda$$

and

$$\int_{-1}^{0}\left(\lambda+1\right)^n\frac{\left(\frac{\lambda}{\lambda+1}\right)^{n+1}-1}{\frac{\lambda}{\lambda+1}-1}d\lambda=\int_{-1}^{0}\left(\left(\lambda+1\right)^{n+1}-\lambda^{n+1}\right)d\lambda.$$

Thus, we get

$$\int_{-1}^{0}\sum_{j=0}^{n}\mathfrak{M}_{j,n}(\lambda)d\lambda=\frac{1+(-1)^n}{n+2}. \tag{29}$$

Combining Eqs. (28) with (29), after some calculations, we get the desired theorem. □

Theorem 4 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} (-1)^{j+n+1} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \mathfrak{C}_{\mathfrak{n}}. \tag{30}$$

Proof By integrating both sides of Eq. (18) with respect to α from -1 to 0, we obtain:

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} (-1)^{j} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \int_{-1}^{0} (\alpha)^{(n)} d\alpha,$$

where $(-\alpha)^{(n)} = (-1)^n (\alpha)_{(n)}$.

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} (-1)^{j+n+1} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \int_{0}^{1} (\alpha)_{(n)} d\alpha, \tag{31}$$

where $\mathfrak{C}_{\mathfrak{n}}$ is the Cauchy numbers of the first kind as follows:

$$\mathfrak{C}_{\mathfrak{n}} = \int_{0}^{1} (\alpha)_{(n)} d\alpha \tag{32}$$

(*cf.* [15]). Combining Eqs. (31) with (32), after some calculations, we get the desired theorem. □

Theorem 5 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} \sum_{k=0}^{n} (-1)^{j+n+1} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \frac{S_1(n,k)}{k+1}. \tag{33}$$

Proof Combining Eqs. (15) with (31), we get the following equation:

$$\int_{-1}^{0} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} (-1)^{j+n+1} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \sum_{k=0}^{n} S_1(n,k) \int_{0}^{1} \alpha^k d\alpha.$$

After the necessary calculations are done in the above equation, we arrive at the assertion of the theorem. □

Combining Eqs. (30) with (33), we get the following known result:

Corollary 4 *Let $n \in \mathbb{N}_0$. Then, we have*

$$\mathfrak{C}_{\mathfrak{n}} = \sum_{k=0}^{n} \frac{S_1(n,k)}{k+1}. \tag{34}$$

There are many different proof of Eq. (34). For instance (Refs. [15, 19]).

Corollary 5 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{0}^{1} \mathfrak{q}_n(\lambda;\alpha)\, d\alpha = \sum_{j=1}^{n} (-1)^{j} S_2(n,j) \mathfrak{M}_{j,n}(\lambda) \hat{C}_n. \tag{35}$$

Proof By integrating both sides of Eq. (18) with respect to α from 0 to 1, we obtain:

$$\int_0^1 \mathfrak{q}_n(\lambda;\alpha)\,d\alpha = \sum_{j=1}^{n}(-1)^{j+n+1} S_2(n,j)\mathfrak{M}_{j,n}(\lambda)\int_0^1 (\alpha)^{(n)} d\alpha, \tag{36}$$

where $\hat{C}_n$ is the Cauchy numbers of the second kind as follows:

$$\hat{C}_n = \int_0^1 (\alpha)^{(n)} d\alpha \tag{37}$$

(*cf.* [15]). □

Combining Eqs. (2) with (10), after some calculations, we get

$$(1+\lambda)^{\alpha} x^n = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}^{(-\alpha)}(\lambda)\, H_k^{(\alpha)}\left(x;-\frac{1}{\lambda}\right).$$

By integrating both sides of the above equation with respect to x from -1 to 0, we obtain:

$$\int_{-1}^{0}(1+\lambda)^{\alpha} x^n dx = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}^{(-\alpha)}(\lambda)\int_{-1}^{0} H_k^{(\alpha)}\left(x;-\frac{1}{\lambda}\right)dx,$$

where

$$H_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right) = \sum_{k=0}^{n}\binom{n}{k}x^k H_{n-k}^{(\alpha)}\left(-\frac{1}{\lambda}\right)$$

(*cf.* [31]). After some necessary calculations in the above equation, we get the following corollary:

Corollary 6 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$(-1)^n\frac{(1+\lambda)^{\alpha}}{n+1} = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}^{(-\alpha)}(\lambda)\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{1}{m+1}H_{k-m}^{(\alpha)}\left(-\frac{1}{\lambda}\right).$$

Theorem 6 *Let $n \in \mathbb{N}_0$. Then, we have*

$$u_n(x;-\lambda) = \left(\lambda - \frac{1}{n!}\right)\frac{\mathcal{B}_{n+1}(\lambda)}{n+1}.$$

Proof Putting Eq. (3), $\lambda \to -\lambda$, we have

$$\frac{1}{t}\left(\frac{t\lambda}{\lambda e^t - 1} - \frac{t}{\lambda e^t - 1}\right) = \sum_{n=0}^{\infty} u_n(x;-\lambda)\frac{t^n}{n!}.$$

By using Eq. (6), we get

$$\lambda\sum_{n=0}^{\infty}\mathcal{B}_n\frac{t^n}{(n+1)\,n!} - \sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n}{(n+1)\,n!} = \sum_{n=0}^{\infty}u_n(x;-\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Table 1 Comparison between exact integral and asymptotic estimate for $\int_{-1}^{0} \mathfrak{q}_n^{(\alpha)}(\lambda)\, d\lambda$

n	Exact integral	Asymptotic estimate	Absolute error
10	198009425.454545	192038761.701381	5970663.75316486
100	9.3505×10^{247}	9.3137×10^{247}	3.6768×10^{245}

All computations were performed using Mathematica® [Wolfram Research Inc., Mathematica Online (Wolfram Cloud), Champaign, IL, USA, 2020]

3 Approximation formula for Eq. (27)

Here by using the Stirling approximation method, we derive approxiamtion formula for integral given in Eq. (27). We set

$$R_{n,j} = \frac{\Gamma(j+1)\,\Gamma(n-j+1)}{\Gamma(n+2)}.$$

By using the following asymptotic forms and using the Stirling approximation:

$$\Gamma(k+1) \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}, \quad \text{as} \quad k \to \infty,$$

we obtain the following asymptotic estimate:

$$R_{n,j} \sim \sqrt{2\pi} \frac{j^{j+\frac{1}{2}} (n-j)^{n-j+\frac{1}{2}}}{(n+1)^{n+\frac{3}{2}}} e.$$

Utilizing the properties of the Stirling approximation, Corollary 3 is reduced to the following theorem:

Theorem 7 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\int_{-1}^{0} \mathfrak{q}_n^{(\alpha)}(\lambda) d\lambda \sim \sum_{j=1}^{n} S_2(n,j)(\alpha)_{(n)} \sqrt{2\pi} \frac{j^{j+\frac{1}{2}} (n-j)^{n-j+\frac{1}{2}}}{(n+1)^{n+\frac{3}{2}}} e.$$

This approximation is valid when both $j \to \infty$ and $n - j \to \infty$ as $n \to \infty$, ensuring the applicability of Stirling's formula (1) Table 1.

4 Applications of Laplace and Mellin transforms to the function $q_n(\lambda; \alpha)$

In this section, we apply the Laplace and Mellin transforms to the function $\mathfrak{q}_n(\lambda; \alpha)$, we give some novel finite sums formulas, which are connected with special functions such as the gamma and beta functions.

By applying the Laplace transform and Mellin transform to the function $\mathfrak{q}_n(\lambda; \alpha)$, respectively, we get

$$\mathcal{L}\{\mathfrak{q}_n(\lambda; \alpha); s\} = \int_0^{\infty} e^{-s\lambda} \mathfrak{q}_n(\lambda; \alpha) d\lambda \tag{38}$$

and

$$\mathcal{M}\{q_n(\lambda; \alpha); s\} = \int_0^{\infty} \lambda^{s-1} q_n(\lambda; \alpha) d\lambda. \tag{39}$$

By combining Eq. (38) with the following Tricomi confluent hypergeometric function (also known as the second kind):

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}\,t^{a-1}(1+t)^{b-a-1}dt, \tag{40}$$

for $\Re(a)>0$, ([17]), we arrive at the following theorem:

Theorem 8 *Let $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ and $\Re(s)>0$. Then, the Laplace transform of $\mathfrak{q}_n(\lambda;\alpha)$ is given by*

$$\mathcal{L}\{\mathfrak{q}_n(\lambda;\alpha);s\}=(\alpha)^{(n)}\sum_{j=0}^{n}(-1)^j S_2(n,j)\,j!\,U(j+1,n+2;s). \tag{41}$$

Proof Applying the Laplace transform, we have

$$\mathcal{L}\{\mathfrak{q}_n(\lambda;\alpha);s\}=(\alpha)^{(n)}\sum_{j=0}^{n}(-1)^j S_2(n,j)\int_0^\infty e^{-s\lambda}\lambda^j(\lambda+1)^{n-j}d\lambda.$$

Substituting $a=j+1$, $b=n+2$ and $z=s$ into Eq. (40), we get

$$U(j+1,n+2;s)=\frac{1}{\Gamma(j+1)}\int_0^\infty e^{-s\lambda}\lambda^j(\lambda+1)^{n-j}d\lambda.$$

After the necessary calculations are done in the above equation, we arrive at the assertion of the theorem. □

By combining Eq. (39) with beta and gamma function, we arrive at the following theorem:

Theorem 9 *Let $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ and $\Re(s)>0$. Then, the Mellin transform of $\mathfrak{q}_n(\lambda;\alpha)$ is given by*

$$\mathcal{M}\{\mathfrak{q}_n(\lambda;\alpha);s\}=(\alpha)^{(n)}\sum_{j=0}^{n}(-1)^j S_2(n,j)\frac{\Gamma(s+j)\Gamma(-n-s+1)}{\Gamma(j-n+1)}, \tag{42}$$

for $\Re(s+j)>0$ and $\Re(n+s)<1$.

Proof Using the definition of the Mellin transform:

$$\mathcal{M}\{\mathfrak{q}_n(\lambda;\alpha);s\}=(\alpha)^{(n)}\sum_{j=0}^{n}(-1)^j S_2(n,j)\int_0^\infty \lambda^{s-1+j}(\lambda+1)^{n-j}\,d\lambda.$$

Applying the identity from beta function and gamma function representation (*cf.* [17]):

$$\int_0^\infty \lambda^{s-1+j}(\lambda+1)^{n-j}d\lambda=B(s+j,-n-s+1)=\frac{\Gamma(s+j)\Gamma(-n-s+1)}{\Gamma(j-n+1)}.$$

Substituting into the main series, we obtain the desired result of the theorem. □

5 New families of Apostol-type splines with special polynomials and numbers

In this section, we give some new definitions by using B-spline and Apostol-type polynomials. In order to define these splines, we begin with the fundamental properties of both B-splines and spline smoothness with Apostol-type polynomials.

We define the Apostol-type spline Frobenius polynomials of order n as follows:

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \int_0^1 N_{0,n}(t) H_n^{(\alpha)}(x-t;\lambda)\, dt. \tag{43}$$

Is it possible to construct generating function for the polynomials $\mathfrak{A}_n^{(\alpha)}(x;\lambda)$? That is,

$$\mathcal{G}_{\mathfrak{A}}(x,t;\lambda) = \sum_{n=0}^{\infty} \mathfrak{A}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}?$$

Theorem 10 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m H_{n-m}^{(\alpha)}(x;\lambda) \int_0^1 N_{0,n}(t) t^m dt. \tag{44}$$

Proof By using the identity for the following shifted Apostol-type polynomials:

$$H_n^{(\alpha)}(x-t;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-t)^m H_{n-m}^{(\alpha)}(x;\lambda) \tag{45}$$

(*cf.* [31]). By combining Eqs. (43) with (45), we arrive at the assertion of the theorem.

Combining Eq. (44) with the following known formula, proved by Goldman [6],where the interval is in the range [0, 1]:

$$N_{0,n}(t) = \frac{t^n}{n!}. \tag{46}$$

□

Therefore, we obtain the following theorem:

Theorem 11 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m H_{n-m}^{(\alpha)}(x;\lambda)}{(n+m+1)n!}. \tag{47}$$

To provide a visual comparison between Apostol-type spline Frobenius polynomials and their classical counterparts, we consider the special case $\lambda = 1$ and $\alpha = \frac{1}{2}$. Figure 1 depicts the graphs of $\mathfrak{A}_n^{(1/2)}(x;1)$ for $n = 0, 1, 2, 3$ on the interval $x \in [-1, 1]$. The figure clearly illustrates that Apostol-type splines exhibit modified curvature and amplitude behavior compared to the classical spline case, reflecting the influence of the Apostol parameter α. This example confirms that the theoretical construction leads to genuinely new spline families.

Now, we compute the integral moments of the uniform B-spline defined over the interval [0, 1] by

$$\mu_m = \int_0^1 N_{0,1}(t) t^m dt.$$

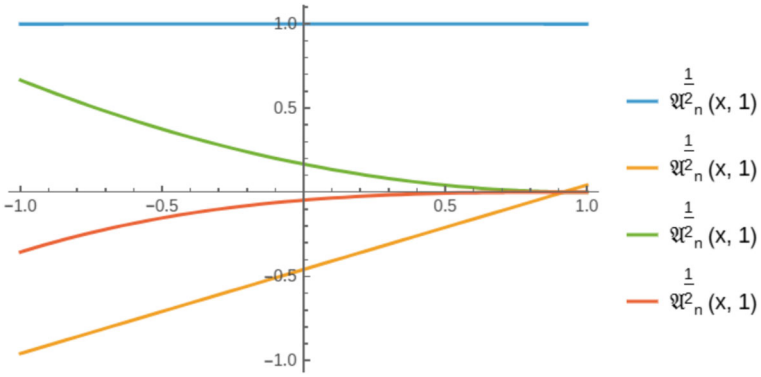

Fig. 1 Graphs of $\mathfrak{A}_n^{(1/2)}(x; 1)$ for $n = 0, 1, 2, 3$ on $[-1, 1]$

We now list the computed values for several values of m, for $m = 0$:

$$\mu_0 = \int_0^1 t dt = \frac{1}{2},$$

for $m = 1$:

$$\mu_1 = \int_0^1 t t dt = \frac{1}{3},$$

for $m = 2$:

$$\mu_2 = \int_0^1 t t^2 dt = \frac{1}{4},$$

for $m = 3$:

$$\mu_3 = \int_0^1 t t^3 dt = \frac{1}{5}.$$

Then, for the interval [0, 1], the integral takes the form:

$$\int_0^1 N_{0,1}(t)\, t^m\, dt = \frac{1}{m+2}. \tag{48}$$

By combining Eqs. (44) with (48), we get the following corollary:

Corollary 7 *Let $0 \le t \le 1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{A}_1^{(\alpha)}(x; \lambda) = \sum_{m=0}^{1} \frac{(-1)^m H_{1-m}^{(\alpha)}(x; \lambda)}{m+2}.$$

Combining Eq. (47) with the following known formula, proved by Gun and Simsek [9]:

$$H_n^{(\alpha)}\left(-\frac{1}{\lambda}\right) = \frac{\mathfrak{q}_n(\lambda; \alpha)}{(1+\lambda)^n},$$

we obtain the following corollary:

Corollary 8 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{A}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{\mathfrak{q}_{n-m}\left(-\frac{1}{\lambda}; \alpha\right)}{(\lambda-1)^{n-m}} \frac{(-1)^m (\lambda)^{n-m}}{(n+m+1)n!}.$$

Substituting $x = 0$ into Eq. (43), we get

$$\mathfrak{A}_n^{(\alpha)}(\lambda) = \int_0^1 N_{0,n}(t) H_n^{(\alpha)}(-t;\lambda)\, dt. \tag{49}$$

Combining Eqs. (46) with (49), we get

$$\sum_{n=0}^{\infty} \mathfrak{A}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\sum_{n=0}^{\infty} H_n^{(\alpha)}(-t;\lambda) \frac{t^n}{n!} \right) dt,$$

we obtain the following corollary:

Corollary 9 *Let $0 \le t \le 1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\sum_{n=0}^{\infty} \mathfrak{A}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\frac{1-\lambda}{e^t - \lambda} \right)^{\alpha} e^{-t^2}\, dt.$$

Theorem 12 *Let $p \le t \le p+1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we get*

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \sum_{j=0}^{p} \sum_{k=0}^{n} \binom{n}{m} \binom{n+1}{j} \binom{n}{k} (-1)^{m+j+n-k} H_{n-m}^{(\alpha)}(x;\lambda) \frac{j^{n-k}}{n!(m+k+1)}.$$

Proof Combining Eqs. (22) with (44), we get the following result:

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m H_{n-m}^{(\alpha)}(x;\lambda) \int_0^1 \left(\frac{1}{n!} \sum_{j=0}^{p} (-1)^j \binom{n+1}{j} (t-j)^n \right) t^m dt.$$

Here, we give integral formula for Eq. (5) as follows: Let

$$I_t = \int_0^1 (t-j)^n t^m dt.$$

Then the following identity holds:

$$I_t = j^{n+m+1} \int_0^{1/j} (u-1)^n u^m\, du,$$

we get

$$\int_0^{1/j} (u-1)^n u^m\, du = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^{1/j} u^{m+k}\, du.$$

Therefore, we get

$$I_t = \sum_{k=0}^{n} \binom{n}{k} \frac{(-j)^{n-k}}{(m+k+1)}. \tag{50}$$

Substituting Eq. (50) into Eq. (5), we arrive at the assertion of the theorem. □

When $j = 1$, we get

$$I_t = \int_0^1 (t-1)^n t^m\, dt = (-1)^n \beta(m+1, n+1). \tag{51}$$

Combining Eq. (50) with Eq. (51), we get the following result:

Corollary 10 *Let $n \in \mathbb{N}_0$. Then, we have*

$$\beta(m+1, n+1) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(m+k+1)}.$$

We define the Apostol-type spline Bernoulli polynomials of order n as follows:

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda) = \int_0^1 N_{0,n}(t) \mathcal{B}_n^{(\alpha)}(x - t; \lambda) dt. \tag{52}$$

Theorem 13 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{B}_{n-m}^{(\alpha)}(x; \lambda) \int_0^1 N_{0,n}(t) t^m dt. \tag{53}$$

Proof By using the identity for the following shifted Apostol-type polynomials:

$$\mathcal{B}_n^{(\alpha)}(x - t; \lambda) = \sum_{m=0}^{n} \binom{n}{m} (-t)^m \mathcal{B}_{n-m}^{(\alpha)}(x; \lambda). \tag{54}$$

By combining Eqs. (52) with (54), we arrive at the assertion of the theorem.

Combining Eq. (46) with the following known formula, proved by Goldman [6],where the interval is in the range [0, 1]: □

Corollary 11 *Let $0 \le t \le 1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m \mathcal{B}_{n-m}^{(\alpha)}(x; \lambda)}{(n+m+1)n!}.$$

Remark 1 For $\lambda = 1$, the Frobenius–Euler polynomials reduce to the generalized Bernoulli polynomials. Consequently, the Apostol-type spline $\mathfrak{A}_n^{(\alpha)}(x; \lambda)$ coincides with the Bernoulli-type spline, which confirms the consistency of our construction.

Substituting $x = 0$ into Eq. (53), we get

$$\mathfrak{B}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{B}_{n-m}^{(\alpha)}(\lambda) \int_0^1 N_{0,n}(t) t^m dt. \tag{55}$$

Combining Eq. (55) with the following known formula, proved by Gun and Simsek [9]:

$$\mathcal{B}_n^{(\alpha)}(-\lambda) = \frac{(-1)^\alpha n! \mathfrak{q}_n(\lambda; \alpha)}{(n-\alpha)!(1+\lambda)^n},$$

we obtain the following corollary:

Corollary 12 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{B}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^{\alpha+m}(n-m)! \mathfrak{q}_n(-\lambda; \alpha)}{(n-\alpha-m)!(1-\lambda)^{\alpha+n-m}} \int_0^1 N_{0,n}(t) t^m dt.$$

Substituting $x = 0$ into Eq. (52), we get

$$\mathfrak{B}_n^{(\alpha)}(\lambda) = \int_0^1 N_{0,n}(t)\mathcal{B}_n^{(\alpha)}(-t;\lambda)dt.$$

Combining Eqs. (7) with (46), we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(-t;\lambda)\frac{t^n}{n!}\right)dt,$$

we obtain the following corollary:

Corollary 13 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{-t^2}\, dt.$$

Theorem 14 *Let $p \le t \le p+1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we get*

$$\mathfrak{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n}\sum_{j=0}^{p}\sum_{k=0}^{n}\binom{n}{m}\binom{n+1}{j}\binom{n}{k}(-1)^{m+j}\mathcal{B}_{n-m}^{(\alpha)}(x;\lambda)\frac{(-j)^{n-k}}{n!(m+k+1)}.$$

Proof Combining Eqs. (22) with (53), we get the following result:

$$\mathfrak{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n}\binom{n}{m}(-1)^m\mathcal{B}_{n-m}^{(\alpha)}(x;\lambda)\frac{1}{n!}\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}\int_0^1 (t-j)^n t^m dt.$$

After the necessary calculations are done in the above equation, we arrive at the assertion of the theorem. □

We define the Apostol-type spline Euler polynomials of order n as follows:

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \int_0^1 N_{0,n}(t)\mathcal{E}_n^{(\alpha)}(x-t;\lambda)\, dt. \tag{56}$$

Theorem 15 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n}\binom{n}{m}(-1)^m\mathcal{E}_{n-m}^{(\alpha)}(x;\lambda)\int_0^1 N_{0,n}(t)t^m dt. \tag{57}$$

Proof By using the identity for the following shifted Apostol-type polynomials:

$$\mathcal{E}_n^{(\alpha)}(x-t;\lambda) = \sum_{m=0}^{n}\binom{n}{m}(-t)^m\mathcal{E}_{n-m}^{(\alpha)}(x;\lambda). \tag{58}$$

By combining Eqs. (56) with (58), we arrive at the assertion of the theorem. □

Combining Eq. (46) with the following known formula, proved by Goldman [6], where the interval is in the range [0, 1]:

Corollary 14 *Let $0 \le t \le 1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m\mathcal{E}_{n-m}^{(\alpha)}(x;\lambda)}{(n+m+1)n!}.$$

Remark 2 For certain values of the parameter λ, the Euler-type spline $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ exhibits behavior similar to the Bernoulli and Frobenius-type cases, especially for small n.

Substituting $x = 0$ into Eq. (57), we get

$$\mathfrak{E}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{E}_{n-m}^{(\alpha)}(\lambda) \int_0^1 N_{0,n}(t) t^m dt. \tag{59}$$

Combining Eq. (59) with the following known formula, proved by Gun and Simsek [9]:

$$\mathcal{E}_n^{(\alpha)}(\lambda) = 2^{\alpha} \frac{\mathfrak{q}_n(\lambda;\alpha)}{(1+\lambda)^{\alpha+n}},$$

we obtain the following corollary:

Corollary 15 *Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{E}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m 2^{\alpha} \frac{\mathfrak{q}_{n-m}(\lambda;\alpha)}{(1+\lambda)^{\alpha+n-m}} \int_0^1 N_{0,n}(t) t^m dt.$$

Substituting $x = 0$ into Eq. (56), we get

$$\mathfrak{E}_n^{(\alpha)}(\lambda) = \int_0^1 N_{0,n}(t) \mathcal{E}_n^{(\alpha)}(-t;\lambda) dt. \tag{60}$$

Combining Eqs. (9) with (46), we have

$$\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(-t;\lambda) \frac{t^n}{n!} \right) dt,$$

we obtain the following corollary:

Corollary 16 *Let $0 \le t \le 1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(\lambda) = \int_0^1 \left(\frac{2}{\lambda e^t + 1} \right)^{\alpha} e^{-t^2} dt.$$

Theorem 16 *Let $p \le t \le p+1$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we get*

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \sum_{j=0}^{p} \sum_{k=0}^{n} \binom{n}{m} \binom{n+1}{j} \binom{n}{k} \mathcal{E}_{n-m}^{(\alpha)}(x;\lambda) \frac{(-1)^{m+j+n-k} j^{n-k}}{n!(m+k+1)}.$$

Proof Combining Eqs. (22) with (57), we get the following result:

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{E}_{n-m}^{(\alpha)}(x;\lambda) \int_0^1 \frac{1}{n!} \sum_{j=0}^{p} (-1)^j \binom{n+1}{j} (t-j)^n t^m dt.$$

After the necessary calculations are done in the above equation, we arrive at the assertion of the theorem. □

6 A unified construction for Apostol-type spline polynomials via cardinal B-splines

In this section, we establish a unified analytical numerical approach for computing Apostol-type spline polynomials by identifying the B-spline used in their definitions with the cardinal B-spline weight functions introduced in Milovanović [16].

Let $\phi_n(x)$ denote the cardinal B-spline of order n, defined over the interval $[0, n]$ with uniformly spaced knots. In Milovanović's construction, these functions are used as weight functions to generate orthogonal polynomial families through moment methods, and to derive highly accurate Gaussian quadrature formulas.

On the other hand, in our formulation of Apostol-type spline polynomials, the integrals are given by uniform B-splines $N_{0,n}(x)$, defined on $[0, n+1]$. These satisfy the same recurrence relation and support structure as the $\phi_{n+1}(x)$.

Thus, we identify:

$$\phi_n(x) = N_{0,n-1}(x), \quad x \in [0, n], \quad n \in \mathbb{N}.$$

This equivalence allows us to apply the symbolic and numerical techniques developed for $\phi_n(x)$ to our spline-based Apostol polynomials $\mathfrak{A}_n^{(\alpha)}(x;\lambda)$, $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$, and $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$.

Combining Eq. (44) with the following known formula, proved by Milovanović's [16], where the interval is in the range $[0, n+1]$:

$$\mu_m = \int_0^1 N_{0,n}(t)t^m,$$

thus, we obtain the following theorem:

Theorem 17 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{A}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m H_{n-m}^{(\alpha)}(x;\lambda)\mu_m.$$

Thanks to the identification $N_{0,n}(t) = \phi_{n+1}(t)$, we may compute the moments μ_m. Symbolically, using exact formulas (e.g., Schoenberg expansions or recurrence methods), or numerically, using Gaussian quadrature rules with respect to $\phi_{n+1}(t)$ as the weight function.

Similarly, for the Apostol-type Bernoulli spline polynomials, by using Eq. (55), we obtain the following theorem:

Theorem 18 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{B}_n^{(\alpha)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{B}_{n-m}^{(\alpha)}(\lambda)\mu_m.$$

For the Apostol-type Euler spline polynomials, by using Eq. (57), we obtain the following theorem:

Theorem 19 *Let $x \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then, we have*

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \mathcal{E}_{n-m}^{(\alpha)}(x;\lambda)\mu_m.$$

We again obtain analogous moment expansions by shifting and expanding the Apostol-type polynomials.

Table 2 Comparison between $\phi_n(x)$ and $N_{0,n-1}(x)$

Order n	Weight function $\phi_n(x)$	Uniform B-spline $N_{0,n-1}(x)$	Support
1	$\phi_1(x) = 1$	$N_{0,0}(x) = 1$	$[0, 1]$
2	$\phi_2(x)$ (linear)	$N_{0,1}(x)$ (linear)	$[0, 2]$
3	$\phi_3(x)$ (quadratic)	$N_{0,2}(x)$ (quadratic)	$[0, 3]$
4	$\phi_4(x)$ (cubic)	$N_{0,3}(x)$ (cubic)	$[0, 4]$
n	$\phi_n(x)$: cardinal B-spline	$N_{0,n-1}(x)$: uniform B-spline	$[0, n]$

The moment sequences $\mu_k = \int_0^1 N_{0,n}(t)t^k dt$ remain the same, and hence the orthogonal polynomial basis and Gaussian quadrature weights computed for $\phi_{n+1}(t)$ apply similarly for all three families.

Table 2 provides a comparative summary between the cardinal B-spline functions $\phi_n(x)$, used in the orthogonal polynomial construction of Milovanović [16], and the uniform B-spline basis functions $N_{0,n-1}(x)$. It illustrates their order, functional form, and respective support intervals, emphasizing their equivalence under uniform knot placement.

7 Conclusion

In this paper, we defined a new family of spline-based polynomials constructed using Apostol-type generating functions and integral representations. In particular, we gave explicit integral formulas and generating functions for Apostol-type spline Bernoulli, Euler, and Frobenius-type polynomials by combining B-spline structures with Apostol-type polynomials. We constructed these new families through moment expansions involving uniform B-splines and provided integral representations based on generating function identities.

We derived several explicit formulas for these spline-based polynomials by employing moment integrals of the form

$$\mu_m = \int_0^1 N_{0,n-1}(t)\, t^m\, dt,$$

where $N_{0,n-1}(t)$ denotes the uniform B-spline of degree $n-1$. These formulas allowed us to express the Apostol-type spline polynomials as expansions involving Apostol polynomials weighted by spline moments. Additionally, we implemented numerical evaluations of these integrals using a Gaussian developed for cardinal B-spline weight functions. The quadrature rules relied on Milovanović's orthogonalization techniques and allowed computation of moment sequences. Moreover, we presented Laplace and Mellin transforms of the integrals. A comparative numerical table confirmed the validity and accuracy of our proposed asymptotic approximations.

We also examined the relationships between the $\mathfrak{q}_n(\lambda;\alpha)$ polynomial structures and combinatorial sequences, such as Stirling numbers of first and second kinds. This work creates a bridge between spline approximation theory and the theory of special polynomials. We may form as a foundation for the construction of new spline-based models in approximation theory, numerical analysis, and mathematical physics. Our results establish new connections between spline theory and special functions, especially through structures involving integrals.

The approach presented here opens new possibilities for further studies in analytic number theory and spline-based approximation methods.

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Data Availability There is no data available.

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