On the Focal Curvatures of Non-lightlike Curves in Minkowski $(m+1)-$Space

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Abstract

In this paper, we show that the focal curvatures of a non-lightlike curve in Minkowski $(m+1)$-space satisfy "scalar Frenet equations". And using the focal curvatures we give necessary and sufficient conditions for a point of a non-lightlike curve to be a vertex.

Keywords : Focal Curve, Focal Curvature, Osculating Hypersphere, Vertex.

Minkowski $(m+1)$-Uzayında Non-Lightlike Eğrilerin Fokal Eğrilikleri Üzerine

Özet :
Bu makalede, Minkowski (m+1) uzayında bir non-lightlike eğrinin fokal eğriliklerinin skalar Frenet denklemlerini sağladığını gösterdik ve fokal eğrilikleri kullanarak non-lightlike bir eğrinin bir noktasının köşe noktası olabilmesi için gerekli ve yeterli şartları verdik.

Anahtar Kelimeler : Fokal Eğri, Fokal Eğrilik, Oskülatör Hiperküre, Köşe.

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1 Introduction

The focal curve of a smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^{m+1}$, consists of the centers of its osculating hypersphere. The focal curve can be state as

$$\mathcal{C}_\alpha(t) = (\alpha + c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + \cdots + c_m\mathbf{N}_m)(t)$$

where $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, ..., \mathbf{N}_m\}$ is the Frenet frame of α and $c_1, c_2, ..., c_m$ are smooth functions that we call focal curvatures of α. Vargas found an expression of the curvatures κ_i of α, in terms of the focal curvatures of c_i and show that the focal curvatures satisfy some "scalar Frenet equations" in the Euclidean space [1]. Vargas gave necessary and sufficient conditions for which the radius R_p of the osculating sphere of dimension p be critical for $p = 1, 2, ..., m$ and gave necessary and sufficient conditions for which a point of a curve be a vertex using focal curvatures. In this paper, we studied to find similar relations for non-lightlike curve in the $(m+1)$-dimensional Lorentzian space.

We introduce some basic definitions as non-lightlike curve, order of contact, osculating hypersphere, vertex and focal curve in section 1, and we state theorems and lemmas related focal curvatures and vertex in section 2. In this paper, we will always assume that the derivatives of α order $1, 2, \ldots, m+1$ are linearly independent at any point.

2 Preliminaries

The Minkowski space $\mathbb{E}_1^{m+1}$ is the Euclidean space $\mathbb{E}^{m+1}$ provided with the Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_1v_1 + \sum_{i=2}^{m+1} u_iv_i$$

where $\mathbf{u} = (u_1, u_2, ..., u_{m+1})$ and $\mathbf{v} = (v_1, v_2, ..., v_{m+1})$.

An arbitrary vector be $\mathbf{u} = (u_1, u_2, ..., u_{m+1})$ in $\mathbb{E}_1^{m+1}$ can have one of three Lorentzian characters; it is spacelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, it is timelike if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ or it is null (or lightlike) if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^{m+1}$ is locally spacelike, timelike or null, if all of its velocity vectors are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. The pseudo norm of an arbitrary vector $\mathbf{u} \in \mathbb{E}_1^{m+1}$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}.$$

The sphere of center $C = (b_1, b_2, ..., b_{m+1})$ and radius $r \in \mathbb{R}^+$ in the Minkowski $(m+1)$-space is defined by

$$S(r) = \left\{ a = (a_1, a_2, ..., a_{m+1}) \in \mathbb{E}_1^{m+1} : \langle a - C, a - C \rangle = \pm r^2 \right\}.$$

Let $\alpha : I \to \mathbb{E}_1^{m+1}$ be a non-lightlike curve parametrized by a arclength s. Suppose that $\{\mathbf{N}_0 = \mathbf{T}(s), \mathbf{N}_1(s), ..., \mathbf{N}_m(s)\}$ is Frenet frame and $\kappa_i(s)$ is i-th curvature function at the point $\alpha(s)$. Then Frenet formulas are found as follows,

1. $\mathbf{T}'(s) = \varepsilon_1 \kappa_1(s) \mathbf{N}_1(s)$
2. $\mathbf{N}_i'(s) = -\varepsilon_{i-1}\kappa_i \mathbf{N}_{i-1}(s) + \varepsilon_{i+1}\kappa_{i+1}(s) \mathbf{N}_{i+1}(s)$, for $1 \leq i \leq m-1$
3. $\mathbf{N}_m'(s) = -\varepsilon_m \kappa_m \mathbf{N}_{m-1}(s)$

where $\varepsilon_i = \langle \mathbf{N}_i, \mathbf{N}_i \rangle$ and $\kappa_i = \langle \mathbf{N}_{i-1}', \mathbf{N}_i \rangle$ for $i = 1, 2, ..., m$.

Definition 1 *We say that α and $F^{-1}(0)$ have k-point contact for $\mathbf{T} = \mathbf{T}_0$ provided the function g defined by*

$$g(t) = F(\alpha_1(t), \alpha_2(t), ..., \alpha_{m+1}(t)) = F(\alpha(t))$$

satisfies $g(t_0) = g'(t_0) = ... = g^{(k-1)}(t_0) = 0$, $g^{(k)} \neq 0$. We also say that the order of contact is k. Dropping the condition $g^{(k)} \neq 0$. We say that there is at least k-point contact. This definition means in the former language of geometers that the curve α and $F^{-1}(0)$ "meet at k consecutive points" or that α and $F^{-1}(0)$ "meet at k infinitely close points" [8].

Definition 2 *For $k = 1, 2, ..., m$ a k-osculating sphere at a point of a non-lightlike curve in the Lorentzian space $\mathbb{E}_1^{m+1}$ is a k-dimensional sphere having at least $(k+2)$-point contact with the curve at that point. For $k = m$ we will simply write osculating hypersphere* [1].

Definition 3 *The focal curve of a smooth curve space $\alpha : I \to \mathbb{E}^{m+1}$ in the Euclidean space $\mathbb{E}^{m+1}$ consist of the centers its osculating hyperspheres. The centers of the osculating hypersphere of α are well defined only for the points of α where all curvatures are different from zero. Let $\alpha : I \to \mathbb{E}_1^{m+1}$ be a non-lightlike curve having all its curvature different from zero at any point. The curve*

$$\mathcal{C}_\alpha : t \to \mathcal{C}_\alpha(t) \in \mathbb{E}_1^{m+1}$$

consisting of the centers of the osculating hyperspheres of the curve $\alpha : t \to \alpha(t) \in \mathbb{E}_1^{m+1}$ is called the parameterized focal curve of α. The focal curve can be write as

$$\mathcal{C}_\alpha(t) = (\alpha + c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + \cdots + c_m\mathbf{N}_m)(t)$$

where $c_1, c_2, ..., c_m$ are smooth functions of the parameter of the curve α [1, 2, 4, 9]. For instance, for a spacelike curve with timelike binormal in $\mathbb{E}_1^3$, we have

$$\mathcal{C}_\alpha(t) = \alpha(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) + \frac{-\kappa'(t)}{\kappa^2(t)\tau(t)}\mathbf{B}(t)$$

where $\kappa(t)$, $\tau(t)$ are the curvature and torsion functions and $\mathbf{N}(t)$, $\mathbf{B}(t)$ are the normal and binormal respectively of α at $\alpha(t)$.

Definition 4 *We call the function c_i, the i-th focal curvature of α.*

Definition 5 *A vertex of a curve in $\mathbb{E}^n$ is a point at which the curve has at least $(n+2)$-point contact with its osculating hypersphere* [2, 3].

Definition 6 *A flattening of point a curve α in $\mathbb{E}^n$ is a point at which the order of contact of α with the osculating hyperplane is at least $(n+1)$, whereas at an ordinary point it is n. At the flattening of a curve in $\mathbb{E}^n$, $\kappa_i \neq 0$ for $i = 1, 2, ..., n-1$, $\kappa_n = 0$ and $\kappa_n' \neq 0$. Namely the center of the osculating hypersphere is not defined and we will say that "it is at infinity". Moreover, at this point, derivatives of α of order $(1, 2, ..., n)$ are linearly dependent. For example, the flattenings of a plane curve are their inflexion. The flattenings of a curve in $\mathbb{E}^3$ are those at which the torsion vanishes* [2, 7].

Definition 7 *A symmetry point of a curve in $\mathbb{E}^n$ is a point at which the center of the osculating hypesphere lies in the osculating hyperplane, namely $c_m = 0$* [1].

3 Theorems and Results

Let $\alpha : I \to \mathbb{E}_1^{m+1}$ be a non-lightlike curve whose curvatures $\kappa_1, \kappa_2, ..., \kappa_m$ are different from zero at any point and whose Frenet frame is $\{\mathbf{N}_0 = \mathbf{T}, \mathbf{N}_1, ..., \mathbf{N}_m\}$.

Lemma 1 *Let $\alpha : t \to (\alpha_1(t), \alpha_2(t), ..., \alpha_{m+1}(t))$ be a non-lightlike curve in $\mathbb{E}_1^{m+1}$. The velocity vector $\mathcal{C}'(t)$ of the focal curve of α at t is proportional to the m-th normal vector $\mathbf{N}_m(t)$ of α.*

Proof. *Consider the family of functions $F : \mathbb{R} \times \mathbb{E}_1^{m+1} \to \mathbb{R}$ defined by*

$$F_c(t) = \frac{1}{2} \|\mathcal{C} - \alpha(t)\|^2 .$$

We can write $-F = \langle \alpha, \mathcal{C} \rangle - g - \dfrac{\mathcal{C}^2}{2}$, where $g = \dfrac{\langle \alpha, \alpha \rangle}{2}$. Here, the following system of $(m+1)$ equations defines the focal curve $\mathcal{C}(t)$ of α.

$$\begin{array}{c} \langle \alpha', \mathcal{C}(t) \rangle - g' = 0 \\ \langle \alpha'', \mathcal{C}(t) \rangle - g'' = 0 \\ \vdots \\ \left\langle \alpha^{(m+1)}, \mathcal{C}(t) \right\rangle - g^{(m+1)} = 0. \end{array} \tag{2.1}$$

Differentiating each equation with respect to t, we obtain the following system of equations,

$$\begin{array}{c} \langle \alpha', \mathcal{C}'(t) \rangle + \langle \alpha'', \mathcal{C}(t) \rangle - g'' = 0 \\ \langle \alpha'', \mathcal{C}'(t) \rangle + \langle \alpha''', \mathcal{C}(t) \rangle - g''' = 0 \\ \vdots \\ \left\langle \alpha^{(m+1)}, \mathcal{C}'(t) \right\rangle + \left\langle \alpha^{(m+2)}, \mathcal{C}(t) \right\rangle - g^{(m+2)} = 0. \end{array} \tag{2.2}$$

Combining the i-th equation of system (2.2) *with the* $(i+1)$*-th equation of system* (2.1)*, for* $i = 1, 2, ..., m$*, we obtain*

$$\begin{array}{c} \langle \alpha', \mathcal{C}'(t) \rangle = 0 \\ \langle \alpha'', \mathcal{C}'(t) \rangle = 0 \\ \vdots \\ \left\langle \alpha^{(m)}, \mathcal{C}'(t) \right\rangle = 0. \end{array} \tag{2.3}$$

This means that the velocity vector of the focal curve is orthogonal to osculating hyperplane of α*. Namely,* $\mathcal{C}'(t)$ *is proportional to the* m*-th normal vector* $\mathbf{N}_m$*.* ■

Lemma 2 *A non-flattening point of a curve in* $\mathbb{E}_1^{m+1}$ *is a* ***vertex*** *if and only if the velocity vector of the focal curve is zero.*

Proof. If the point $\alpha(t)$ is a vertex of α, besides the system of equations (2.1), it also satisfy the equation

$$\left\langle \alpha^{(m+2)}, \mathcal{C}(t) \right\rangle - g^{(m+2)} = 0 \tag{2.4}$$

Combining (2.4) with the system of equations (2.2), we obtain the equation

$$\left\langle \alpha^{(m+1)}, \mathcal{C}'(t) \right\rangle = 0$$

The preceding equation together with the system (2.3) imply that for a non-flat vertex $\alpha(t)$ of the curve α, the velocity vector $\mathcal{C}'(t)$ of the focal curve is zero. Conversely, if a point $\alpha(t_0)$ is not a vertex then the corresponding point of the focal curve satisfies the relation

$$\left\langle \alpha^{(m+2)}(t_0), \mathcal{C}(t_0) \right\rangle - g^{(m+2)}(t_0) \neq 0$$

which together with the last equation of (2.2) for $t = t_0$, imply that $\mathcal{C}'(t) \neq 0$. ■

Theorem 1 *The focal curvatures of the non-lightlike curve* $\alpha : I \to \mathbb{E}_1^{m+1}$*, parameterized by arc length* s*, satisfy the following "scalar Frenet equations" for* $c_m \neq 0$*.*

$$\begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 c_1' \\ \varepsilon_2 c_2' \\ \vdots \\ \varepsilon_{m-2} c_{m-2}' \\ \varepsilon_{m-1} c_{m-1}' \\ \varepsilon_m c_m' - \dfrac{(R_m^2)'}{2c_m} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & \dots & 0 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \dots & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \kappa_{m-1} & 0 \\ 0 & 0 & 0 & \dots & -\kappa_{m-1} & 0 & \kappa_m \\ 0 & 0 & 0 & \dots & 0 & -\kappa_m & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ \vdots \\ c_{m-2} \\ c_{m-1} \\ c_m \end{bmatrix}$$

Proof. Let $\mathcal{C}_\alpha(s) = (\alpha + c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + \cdots + c_m\mathbf{N}_m)(s)$ be the focal curve of α. Applying Frenet equations to derivative of $\mathcal{C}_\alpha(s)$ with respect to the arc length of α, we obtain,

$$\begin{aligned}
\mathcal{C}'_\alpha(s) &= \alpha' + c'_1\mathbf{N}_1 + c_1(-\varepsilon_0\kappa_1\mathbf{T} + \varepsilon_2\kappa_2\mathbf{N}) + c'_2\mathbf{N}_2 \\
&+ c_2(-\varepsilon_1\kappa_2\mathbf{N}_1 + \varepsilon_3\kappa_3\mathbf{N}_3) + \cdots + c'_{m-1}\mathbf{N}_{m-1} \\
&+ c_{m-1}\left(-\varepsilon_{(m-2)}\kappa_{(m-1)}\mathbf{N}_{(m-2)} + \varepsilon_m\kappa_m\mathbf{N}_m\right) \\
&+ c'_m\mathbf{N}_m + c_m\left(-\varepsilon_{(m-1)}\kappa_{(m-1)}\mathbf{N}_{(m-1)}\right) \\
&= \mathbf{T}(1 - \varepsilon_0 c_1\kappa_1) + \mathbf{N}_1(c'_1 - c_2\varepsilon_1\kappa_2) + \mathbf{N}_2(c'_2 + c_1\varepsilon_2\kappa_2 - c_3\varepsilon_2\kappa_3) \\
&+ \cdots + \mathbf{N}_i\left(c'_i + c_{(i-1)}\varepsilon_i\kappa_i - c_{(i+1)}\varepsilon_i\kappa_{(i+1)}\right) + \cdots + \mathbf{N}_m\left(c'_m + c_{(m-1)}\varepsilon_m\kappa_m\right).
\end{aligned}$$

Because of that $\mathcal{C}'_\alpha(s)$ is proportional to the m-th normal vector $\mathbf{N}_m$ the first $m-1$ components $\mathcal{C}'_\alpha$ vanish. This implies that,

$$\begin{aligned}
1 &= \varepsilon_0 c_1\kappa_1 \\
c'_1 &= \varepsilon_1 c_2\kappa_2 \\
c'_2 &= (-c_1\kappa_2 + c_3\kappa_3)\varepsilon_2 \\
&\vdots \\
c'_{(m-1)} &= \left(-c_{(m-2)}\kappa_{(m-1)} + c_m\kappa_m\right)\varepsilon_{(m-1)}
\end{aligned} \tag{2.5}$$

Thus we get

$$\mathcal{C}'_\alpha(s) = \mathbf{N}_m\left(c'_m + c_{(m-1)}\varepsilon_m\kappa_m\right).$$

Using the fact that the radius R_m of the osculating hypersphere satisfies $(R_m^2) = \|\mathcal{C}_\alpha - \alpha\|^2$, we obtain

$$\begin{aligned}
(R_m^2)' &= \langle \mathcal{C}_\alpha - \mathbf{T}, \mathcal{C}_\alpha - \alpha\rangle' \\
&= 2\langle \mathcal{C}'_\alpha - \mathbf{T}, \mathcal{C}_\alpha - \alpha\rangle \\
&= 2\left\langle \mathbf{N}_m\left(c'_m + \varepsilon_m c_{(m-1)}\kappa_m\right) - \mathbf{T}, c_1\mathbf{N}_1 + c_2\mathbf{N}_2 + ... + c_m\mathbf{N}_m\right\rangle \\
&= 2c_m\varepsilon_m\left(c'_m + \varepsilon_m c_{(m-1)}\kappa_m\right).
\end{aligned} \tag{2.6}$$

Thus, for $c_m \neq 0$, $\varepsilon_m c'_m - \dfrac{(R_m^2)'}{2c_m} = -c_{(m-1)}\kappa_m$. This equation together with (2.5) proves the Theorem 1. ■

Remark 1 *Since radius of osculating sphere of a spherical curve is constant, the last component of the left hand side vector is just $\varepsilon_m c'_m$ for spherical non-lightlike curves in the Theorem 1.*

Theorem 2 *A non-flattening point of a non-lightlike curve parameterized by arc length in $\mathbb{E}_1^{m+1}$, is a **vertex** if and only if*

$$c_m' + \varepsilon_m c_{(m-1)}\kappa_m = 0$$

at that point.

Proof. According to Lemma 2, a non flattening of a curve is a vertex is and only if the velocity vector of the focal curve is zero. So $c_m' + \varepsilon_m c_{(m-1)}\kappa_m = 0$ at that point. ■

Corollary 1 *A non-lightlike curve parameterized by arc length in $\mathbb{E}_1^{m+1}$, is spherical if and only if $c_m' + \varepsilon_m c_{(m-1)}\kappa_m = 0$.*

Theorem 4 *The curvature of a non-lightlike curve in $\mathbb{E}_1^{m+1}$, parameterized by arc length, may be obtained in terms of the focal curvatures of α by the formula*

$$\kappa_i = \frac{\varepsilon_1 c_1 c_1' + \varepsilon_2 c_2 c_2' + ... + \varepsilon_{(i-2)} c_{(i-2)} c_{(i-2)}' + \varepsilon_{(i-1)} c_{(i-1)} c_{(i-1)}'}{c_{(i-1)} c_i} \quad for \quad i \leq 2.$$

Proof. The proof will be given by induction. From the Theorem 1, we have that

$$\kappa_1 = \frac{\varepsilon_0}{c_1} \ ;$$
$$\kappa_2 = \frac{c_1'}{\varepsilon_1 c_2} = \frac{\varepsilon_1 c_1 c_1'}{c_1 c_2} \ ;$$
$$\kappa_3 = \frac{\varepsilon_2 c_2' + c_1\kappa_2}{c_3} = \frac{\varepsilon_2 c_2' + c_1 \dfrac{\varepsilon_1 c_1'}{c_2}}{c_3} = \frac{\varepsilon_1 c_1 c_1' + \varepsilon_2 c_2 c_2'}{c_2 c_3}$$

Suppose that

$$\kappa_i = \frac{\varepsilon_1 c_1 c_1' + \varepsilon_2 c_2 c_2' + ... + \varepsilon_{(i-2)} c_{(i-2)} c_{(i-2)}' + \varepsilon_{(i-1)} c_{(i-1)} c_{(i-1)}'}{c_{(i-1)} c_i}, \ \text{for} \quad i \leq 2. \quad (2.7)$$

From the (2.5), we know that

$$\varepsilon_i c_i' = c_{(i+1)}\kappa_{(i+1)} + c_{(i-1)}\kappa_i.$$

If we substitute κ_i in this equation, we obtain.

$$\kappa_{(i+1)} = \frac{\varepsilon_1 c_1 c_1' + \varepsilon_2 c_2 c_2' + ... + \varepsilon_{(i-1)} c_{(i-1)} c_{(i-1)}' + \varepsilon_i c_i c_i'}{c_i c_{(i+1)}}, \ \text{for} \ i \leq 2.$$

■

Theorem 5 *The radius R_p of the osculating p-sphere of a non-lightlike curve parameterized by arc length in the Lorentzian space $\mathbb{E}_1^{m+1}$, is critical if and only if*

$$\begin{cases} c_2 = 0 & \text{for } p = 1 \\ c_p = 0 \ \text{ or } \ c_{(i+1)} = 0 & \text{for } 1 < p < m \\ c_m = 0 \ \text{ or } \ c_m' + \varepsilon_m c_{(m-1)} \kappa_m = 0 & \text{for } p = m. \end{cases}$$

Proof. $R_p = \sqrt{\varepsilon_1 c_1^2 + \varepsilon_2 c_2^2 + ... + \varepsilon_p c_p^2}$ is the radius of the osculating p-sphere. Thus, we have

$$R_p R_p' = \varepsilon_1 c_1 c_1' + \varepsilon_2 c_2 c_2' + ... + \varepsilon_p c_p c_p'$$

Applying the formula (2.7), we obtain

$$R_p R_p' = \kappa_{(p+1)} c_{(p+1)} c_p \quad \text{for} \quad 1 \leq p \leq m.$$

For a non-lightlike curve whose derivatives of order $1, 2, ..., m+1$ are linear independent in $\mathbb{E}_1^{m+1}$, $R_p' = 0$ if and only if either $c_p = 0$ or $c_{(p+1)} = 0$ for $1 < p < m$. Moreover, for a smooth curve since $R_p = c_1$ never vanishes, $c_2 = 0$, for $p = 1$. This proves a) and b). And also $p = m$, we have

$$\left(R_m^2\right)' = 2 c_m \varepsilon_m \left(c_m' + \varepsilon_m c_{(m-1)} \kappa_m\right)$$

from the equation (2.6). This proves c). ■

Corollary 3 *If the p-th focal curvature c_p vanishes at a point, then radii R_p and $R_{(p-1)}$ of the osculating p-sphere and $(p-1)$-sphere are critical at that point.*

Corollary 4 *The radius of the osculating hypersphere of a curve at a point is critical if and only if either the point is a vertex or the point is a symmetry point.*

REFERENCES

1. R. Uribe Vargas, "Scalar Frenet Equations" and Focal Curvatures for Curves in $\mathbb{R}^{m+1}$, Institut de Mathematiques de Jussieu, Universites Paris, Preprint, 2000.

2. R. Uribe Vargas, On the Higher Dimensional Four Vertex Theorem, C.R. Acad. Sci. Paris, t.321, Serie I, 1353-1358, 1995.

3. R. Uribe Vargas, On the Twisting of Closed Curves in R" and the Focal Curve, the Darboux Vertices and the Spherical Indicatrices of a Closed Curve in the Euclidean Space $\mathbb{R}^3$, Institut de Mathematiques de Jussieu, Universites Paris, Preprint, 2000.

4. H.H. Hacisalihoglu, Diferensiyel Geometri, Ankara Univ. Fen Fak. Yay., Ankara, 269 s, 1993.

5. E. Özdamar , H.H. Hacisalihoglu, Characterizations of Spherical Curves in Euclidean n-Space, Communications, Serie A, .23, 109-125, 1974.

6. M. Bektaş., M. Ergüt, D. Soylu, The Characterization of the Spherical Timelike Curves in 3-dimensional Lorentzian Space, Bull, of the Malaysian Mathematical Society, 1998.
7. M.C. Romero-Fuster, E. Sanabria-Codesal, Generalized Helices, Twistings and Flattenings of Curves in n-Space, Matematica Contemporonea, Brasil, 17, 267-280, 1999.
8. J.W. Bruce, P.J. Giblin, Curves and Singularities, Cambridge Univ. Press. Second Edition, Cambridge, 321 p, 1992.
9. I. R. Porteous, Geometric Differentiation for the intelligance of curves and surfaces, Cambridge Univ. Press, Cambridge, 301 p, 1995.
10. Petrovic M. Sucurovic E. Some Characterizations of the Lorentzian Spherical Timelike and Null Curves, Matematiki Vesnik, 53, 21-27, 2000.
11. S. Izumiya, D. Pei, T. Sano, The Lightcone Gauss Map and the Lightcone Developable of Spacelike Curve in Minkowski 3-space, Glaskow Math. J., 42, 75-89, 2000.