Adv. Appl. Clifford Algebras 26 (2016), 279–304
© 2015 Springer Basel
0188-7009/010279-26
published online August 22, 2015
DOI 10.1007/s00006-015-0592-3

Advances in Applied Clifford Algebras



An Alternative Approach to Elliptical Motion

Mustafa Özdemir*

Abstract. Elliptical rotation is the motion of a point on an ellipse through some angle about a vector. The purpose of this paper is to examine the generation of elliptical rotations and to interpret the motion of a point on an ellipsoid using elliptic inner product and elliptic vector product. To generate an elliptical rotation matrix, first we define an elliptical ortogonal matrix and an elliptical skew symmetric matrix using the associated inner product. Then we use elliptic versions of the famous Rodrigues, Cayley, and Householder methods to construct an elliptical rotation matrix. Finally, we define elliptic quaternions and generate an elliptical rotation matrix using those quaternions. Each method is proven and is provided with several numerical examples.

Mathematics Subject Classification. 15A63, 15A66, 70B05, 70B10, 70E17, 53A17.

Keywords. Elliptic quaternion, Rotation matrices, Rodrigues formula, Cayley transformation, Householder transformation.

1. Introduction

A rotation is an example of an isometry, a map that moves points without changing the distances between them. A three dimensional rotation is a linear transformation that describes the motion of a rigid body around an axis and can be expressed with an orthonormal matrix which is called a rotation matrix. 3×3 rotation matrices form a special non abelian orthogonal group, denoted by $\mathbf{SO}(3)$. The group of 3×3 rotation matrices is isomorphic to the group of rotations in a 3 dimensional space. This means that multiplication of rotation matrices corresponds to composition of rotations. Rotation matrices are used extensively for computations in geometry, kinematics, physics, computer graphics, animations, and optimization problems involving the estimation of rigid body transformations. For this reason, the generation of a rotation matrix is considered to be an important problem in mathematics.

*Corresponding author.

In the two dimensional Euclidean space, a rotation matrix can easily be generated using basic linear algebra or complex numbers. Similarly, in the Lorentzian plane, a rotation matrix can be generated by double (hyperbolic) numbers. In higher dimensional spaces, obtaining a rotation matrix using the inner product is impractical since each column and row of a rotation matrix must be a unit vector perpendicular to all other columns and rows, respectively. These constraints make it difficult to construct a rotation matrix using the inner product. Instead, in higher dimensional spaces, rotation matrices can be generated using various other methods such as unit quaternions, the Rodrigues formula, the Cayley formula, and the Householder transformation. We will give a brief review of these methods and use elliptical versions of these methods later in the paper.

1. *A unit quaternion*: Each unit quaternion represents a rotation in the Euclidean 3-space. That is, only four numbers are enough to construct a rotation matrix, the only constraint being that the norm of the quaternion is equal to 1. Also, in this method, the rotation angle and the rotation axis can be determined easily. However, this method is only valid in the three dimensional spaces [27,30]. In the Lorentzian space, timelike split quaternions are used instead of ordinary usual quaternions [19,22].
2. *Rodrigues formula*: An orthonormal matrix can be obtained using the matrix exponential $e^{\theta A}$ where A is a skew symmetric matrix and θ is the rotation angle. In this method, only three numbers are needed to construct a rotation matrix in the Euclidean 3-space [2,14,17,24]. The vector set up with these three numbers gives the rotation axis. This method can be extended to the n dimensional Euclidean and Lorentzian spaces [6,11,12,16,20].
3. *Cayley formula*: The formula $C = (I + A)(I - A)^{-1}$ gives a rotation matrix, where A is a skew symmetric matrix. Rotation matrices can be given by the Cayley formula without using trigonometric functions [4,7,18,21,28].
4. *Householder transformation*: The Householder transformation gives us a reflection matrix. We can obtain a rotation matrix using two Householder transformations. This method is an elegant method but it can be long and tedious. Also, the rotation angle has to be calculated separately. This transformation can be used in several scalar product spaces [1,8,15,26,29].

Details about generating rotation matrices, particularly in the Euclidean and Lorentzian spaces, using these methods can be found in various papers, some of which are given in the reference section. Those authors mostly studied the rotation matrices in the positive definite scalar product space whose associated matrices are $\mathrm{diag}(\pm 1, \ldots, \pm 1)$, and interpreted the results geometrically. For example, quaternions and timelike split quaternions were used to generate rotation matrices in the three dimensional Euclidean and Lorentzian spaces where the associated matrices were $diag(1, 1, 1)$ and $diag(-1, 1, 1)$, respectively. In these spaces, rotations occur on the sphere $x^2 + y^2 + z^2 = r^2$

or the hyperboloids $-x^2+y^2+z^2=\pm r^2$. That is, Euclidean and Lorentzian rotation matrices help us to understand spherical and hyperbolic rotations. In the Euclidean space, a rotation matrix rotates a point or a rigid body through a circular angle about an axis. That is, the motion happens on a circle. Similarly, in the Lorentzian space, a rotation matrix rotates a point through an angle about an axis circularly or hyperbolically depending on whether the rotation axis is timelike or spacelike, respectively.

In this paper, we investigate elliptical rotation matrices, which are orthogonal matrices in the scalar product space, whose associated matrix is $\mathrm{diag}(a_1,a_2,a_3)$ with $a_1,a_2,a_3\in\mathbb{R}^+$. First, we choose a proper scalar product to the given ellipse (or ellipsoid) such that this ellipse (or ellipsoid) is equivalent to a circle (or sphere) for the scalar product space. That is, the scalar product doesn't change the distance between any point on the ellipse (or ellipsoid) and origin. Interpreting a motion on an ellipsoid is an important concept since planets usually have ellipsoidal shapes and elliptical orbits. The geometry of ellipsoid can be examined using affine transformations, because of an ellipsoid can be considered as an affine map of the unit sphere. For example, for the ellipsoid $\mathcal{E}^2=\left\{\mathbf{x}\in\mathbb{R}^3:\mathbf{x}^tA\mathbf{x}\leq 1\right\}$ and the unit sphere $S^2=\left\{\mathbf{x}\in\mathbb{R}^3:\|\mathbf{x}\|=\mathbf{x}^t\mathbf{x}\leq 1\right\}$, we can write $\mathcal{E}^2=T(S^2)$ using the affine transformation $T(\mathbf{x})=A\mathbf{x}+c$, $x\in\mathcal{E}^2$. Then we have, $\mathbf{Vol}(\mathcal{E}^2)=\mathbf{Vol}(T(S^2))=\sqrt{\det Q}\mathbf{Vol}(S^2)=\sqrt{\det Q}4\pi/3$ where $Q=AA^t$.

The aim of this study is to explain the motion on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

as a rotation, using the proper inner product, vector product and elliptical orthogonal matrices. In this method, the elliptical inner product, the vector product and the angles are compatible with the parameters θ and β of the parametrization

$$\varphi(\theta,\beta)=(a\cos\theta\cos v,b\cos\theta\sin\beta,c\sin\theta).$$

We use the classical methods to generate elliptical rotation matrices. In the Preliminaries section, first we explain how to define a suitable scalar product and a vector product for a given ellipsoid. Then we introduce the symmetric, skew symmetric and orthogonal matrices in this elliptical scalar product space. Finally, we examine the motion on an ellipsoid using elliptical rotation matrices. In Sect. 3, we generate the elliptical rotation matrices using various classical methods (such as, Cayley formula, Rodrigues formula and Householder transformation) compatible with the defined scalar product. Furthermore, we defined the elliptic quaternions and generate elliptical rotations using unit elliptic quaternions.

2. Preliminaries

We begin with a brief review of scalar products. More informations can be found in [15,25,26]. Consider the map

$$\mathcal{B}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},(\mathbf{u},\mathbf{v})\to\mathcal{B}(\mathbf{u},\mathbf{v})$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. If such a map is linear in each argument, that is,

$$\mathcal{B}(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\mathcal{B}(\mathbf{u}, \mathbf{w}) + b\mathcal{B}(\mathbf{v}, \mathbf{w}),$$
$$\mathcal{B}(\mathbf{u}, c\mathbf{v} + d\mathbf{w}) = c\mathcal{B}(\mathbf{u}, \mathbf{v}) + d\mathcal{B}(\mathbf{u}, \mathbf{w}),$$

where, $a, b, c, d \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then it is called a bilinear form. Given a bilinear form on $\mathbb{R}^n$, there exists a unique $\Omega \in \mathbb{R}^{n\times n}$ square matrix such that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t\Omega\mathbf{v}$. Ω is called "the matrix associated with the form" with respect to the standard basis and we will denote $\mathcal{B}(\mathbf{u}, \mathbf{v})$ as $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{v})$ as needed. A bilinear form is said to be symmetric or skew symmetric if $\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{u})$ or $\mathcal{B}(\mathbf{u}, \mathbf{v}) = -\mathcal{B}(\mathbf{v}, \mathbf{u})$, respectively. Hence, the matrix associated with a symmetric bilinear form is symmetric, and similarly, the associated matrix of a skew symmetric bilinear form is skew symmetric. Also, a bilinear form is nondegenerate if its associated matrix is non-singular. That is, for all $\mathbf{u} \in \mathbb{R}^n$, there exists $\mathbf{v} \in \mathbb{R}^n$, such that $\mathcal{B}(\mathbf{u}, \mathbf{v}) \neq 0$. A real scalar product is a non-degenerate bilinear form. The space $\mathbb{R}^n$ equipped with a fixed scalar product is said to be a real scalar product space. Also, some scalar products, like the dot product, have positive definitely property. That is, $\mathcal{B}(\mathbf{u}, \mathbf{u}) \geq 0$ and $\mathcal{B}(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = 0$. Now, we will define a positive definite scalar product, which we call the $\mathcal{B}$-inner product or elliptical inner product.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$. Then the map

$$\mathcal{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \mathcal{B}(\mathbf{u}, \mathbf{w}) = a_1u_1w_1 + a_2u_2w_2 + \cdots + a_nu_nw_n$$

is a positive definite scalar product. We call it elliptical inner product or $\mathcal{B}$-inner product. The real vector space $\mathbb{R}^n$ equipped with the elliptical inner product will be represented by $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$ or $\mathbb{R}^n_{\mathcal{B}}$. Note that the scalar product $\mathcal{B}(\mathbf{u}, \mathbf{v})$ can be written as $\mathcal{B}(\mathbf{u}, \mathbf{w}) = \mathbf{u}^t\Omega\mathbf{w}$ where associated matrix is

$$\Omega = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_n \end{bmatrix}. \tag{1}$$

The number $\sqrt{\det\Omega}$ will be called "constant of the scalar product" and denoted by Δ in the rest of the paper. The norm of a vector associated with the scalar product $\mathcal{B}$ is defined as $\|\mathbf{u}\|_{\mathcal{B}} = \sqrt{\mathcal{B}(\mathbf{u}, \mathbf{u})}$. Two vectors $\mathbf{u}$ and $\mathbf{w}$ are called $\mathcal{B}$-orthogonal or elliptically orthogonal vectors if $\mathcal{B}(\mathbf{u}, \mathbf{w}) = 0$. In addition, if their norms are 1, then they are called $\mathcal{B}$-orthonormal vectors. If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an $\mathcal{B}$-orthonormal base of $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$, then $\det(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) = \Delta^{-1}$. The cosine of the angle between two vectors $\mathbf{u}$ and $\mathbf{w}$ is defined as,

$$\mathbf{cos}\,\theta = \frac{\mathcal{B}(\mathbf{u}, \mathbf{w})}{\|\mathbf{u}\|_{\mathcal{B}}\|\mathbf{w}\|_{\mathcal{B}}}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid.

Let $\mathcal{B}$ be a non degenerate scalar product, Ω the associated matrix of $\mathcal{B}$, and $R \in \mathbb{R}^{n\times n}$ is any matrix.

(i) If $\mathcal{B}(R\mathbf{u}, R\mathbf{w}) = \mathcal{B}(\mathbf{u}, \mathbf{w})$ for all vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$, then R is called a $\mathcal{B}$-orthogonal matrix. It means that orthogonal matrices preserve the norm of vectors and satisfy the matrix equality $R^t\Omega R = \Omega$. Also, all rows (or columns) are $\mathcal{B}$-orthogonal to each other. We denote the set of $\mathcal{B}$-orthogonal matrices by $\mathbf{O}_{\mathcal{B}}(n)$. That is,

$$\mathbf{O}_{\mathcal{B}}(n) = \{R \in \mathbb{R}^{n\times n} : R^t\Omega R = \Omega \text{ and } \det R = \pm 1\}.$$

$\mathbf{O}_{\mathcal{B}}(n)$ is a subgroup of $\mathbf{Gl}_{\mathcal{B}}(n)$. It is sometimes called the isometry group of $\mathbb{R}^n$ associated with scalar product $\mathcal{B}$. The determinant of a $\mathcal{B}$-orthogonal matrix can be either -1 or 1. If $\det R = 1$, then we call it a $\mathcal{B}$-rotation matrix or an elliptical rotation matrix. If $\det R = -1$, we call it an elliptical reflection matrix. Although the set $\mathbf{O}_{\mathcal{B}}(n)$ is not a linear subspace of $\mathbb{R}^{n\times n}$, it is a Lie group. The isometry group for the bilinear or sesquilinear forms can be found in [15]. The set of the $\mathcal{B}$-rotation matrices of $\mathbb{R}^n$ can be expressed as follows:

$$\mathbf{SO}_{\mathcal{B}}(n) = \{R \in \mathbb{R}^{n\times n} : R^t\Omega R = \Omega \text{ and } \det R = 1\}.$$

$\mathbf{SO}_{\mathcal{B}}(n)$ is a subgroup of $\mathbf{O}_{\mathcal{B}}(n)$.

(ii) If $\mathcal{B}(S\mathbf{u}, \mathbf{w}) = \mathcal{B}(\mathbf{u}, S\mathbf{w})$ for all vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$, then S is called a $\mathcal{B}$-symmetric matrix. It satisfies $S^t\Omega = \Omega S$. The set of $\mathcal{B}$-symmetric matrices, defined by

$$\mathbb{J} = \left\{S \in \mathbb{R}^{n\times n} : \mathcal{B}(S\mathbf{u}, \mathbf{w}) = \mathcal{B}(\mathbf{u}, S\mathbf{w}) \text{ for all } \mathbf{u}, \mathbf{w} \in \mathbb{R}^n\right\}$$

is a Jordan algebra [15]. It is a subspace of the vector space of real $n \times n$ matrices, with dimension $n(n+1)/2$. Any $\mathcal{B}$-symmetric matrix in $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$ can be defined as

$$S = \left[\frac{\Delta a_{ij}}{a_i}\right]_{n\times n} \tag{2}$$

where $a_{ij} = a_{ji}$ and $a_{ij} \in \mathbb{R}$.

(iii) If $\mathcal{B}(T\mathbf{u}, \mathbf{w}) = -\mathcal{B}(\mathbf{u}, T\mathbf{w})$ for all vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$, then T is called a $\mathcal{B}$-skew-symmetric matrix. Also, $T^t\Omega = -\Omega T$. The set of $\mathcal{B}$-skew symmetric matrices, defined by

$$\mathbb{L} = \left\{T \in \mathbb{R}^{n\times n} : \mathcal{B}(T\mathbf{u}, \mathbf{w}) = -\mathcal{B}(\mathbf{u}, T\mathbf{w}) \text{ for all } \mathbf{u}, \mathbf{w} \in V\right\}$$

is a Lie algebra [15]. It is a subspace of the vector space of real $n \times n$ matrices, with dimension $n(n-1)/2$, as well. Any $\mathcal{B}$-skew-symmetric matrix in $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$ can be defined as,

$$T = [t_{ij}]_{n\times n} \quad \text{with } t_{ij} = \begin{cases} \dfrac{\Delta a_{ij}}{a_i} & i > j \\ \dfrac{-\Delta a_{ij}}{a_i} & i < j \\ 0 & i = j \end{cases} \tag{3}$$

where $a_{ij} = a_{ji}$ and $a_{ij} \in \mathbb{R}$.

For example, in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$, the symmetric and skew symmetric matrices are

$$S = \Delta \begin{bmatrix} a_{11}/a_1 & x/a_1 & y/a_1 \\ x/a_2 & a_{22}/a_2 & z/a_2 \\ y/a_3 & z/a_3 & a_{33}/a_3 \end{bmatrix} \text{ and } T = \Delta \begin{bmatrix} 0 & x/a_1 & y/a_1 \\ -x/a_2 & 0 & z/a_2 \\ -y/a_3 & -z/a_3 & 0 \end{bmatrix}.$$

Note that, even if we omit the scalar product constant Δ in S or T, they will still be symmetric or skew symmetric matrix, respectively. But then, we cannot generate elliptical rotation matrices using the Rodrigues and Cayley formulas. So, we will keep the constant Δ.

Now, we define the elliptical vector product, which is related to elliptical inner product. Let $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{in}) \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n-1$ and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be standard unit vectors for $\mathcal{B}$. Then, the elliptical vector product in $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$ is defined as,

$$\begin{aligned} &\mathbb{R}^n_{a_1,a_2,\ldots,a_n} \times \mathbb{R}^n_{a_1,a_2,\ldots,a_n} \times \cdots \times \mathbb{R}^n_{a_1,a_2,\ldots,a_n} \to \mathbb{R}^n_{a_1,a_2,\ldots,a_n}, \\ &(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) \to \mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \cdots \times \mathbf{u}_{n-1}) \\ &\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \cdots \times \mathbf{u}_{n-1}) \\ &\quad = \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 & \cdots & \mathbf{e}_n/a_n \\ u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ u_{21} & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{(n-1)1} & u_{(n-1)2} & u_{(n-1)3} & \cdots & u_{(n-1)n} \end{bmatrix} \end{aligned} \tag{4}$$

The vector $\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \cdots \times \mathbf{u}_{n-1})$ is $\mathcal{B}$-orthogonal to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-1}$ geometrically. As a special case, the 3-dimensional elliptical vector product of $\mathbf{u}_1$ and $\mathbf{u}_2$ is defined as

$$\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2) = \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix}$$

for the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$, where $\Delta = \sqrt{a_1 a_2 a_3}$. The ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$, $a_i \in \mathbb{R}^+$ is the unit sphere for this space. The end point any unit vector in $\mathbb{R}^3_{a_1,a_2,a_3}$ falls on the ellipsoid. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors in $\mathbb{R}^3_{a_1,a_2,a_3}$, then $\mathcal{V}(\mathbf{u}_1 \times \mathbf{u}_2)$ is also unit vector and it is elliptically orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. The standard vector product is a special case of the elliptical vector product. That is, if we take $a_1 = a_2 = a_3 = 1$, we end up standard vector product, standard inner product and, standard orthogonality in the three dimensional Euclidean space.

3. Generating an Elliptical Rotation Matrix

In this section, we will generate elliptical rotation matrices using the elliptical versions of the classical methods. For a given ellipse in the form

$$(\mathbf{E}) : a_1x^2 + a_2y^2 = \lambda, \quad \lambda, a_1, a_2 \in \mathbb{R}^+ \tag{5}$$

we will use the scalar product $\mathcal{B}(\mathbf{u}, \mathbf{w}) = a_1u_1w_1 + a_2u_2w_2$. That is, our scalar product space is $\mathbb{R}^2_{a_1,a_2}$. An elliptical rotation matrix represents a rotation on (**E**) or any ellipse similar to (**E**). Recall that ellipses with the same eccentricity are called similar. Since the shape of an ellipse depends only on the ratio $a_1/a_2, \lambda$ in Eq. (5), does not affect the rotation matrix.

3.1. Rodrigues Rotation Formula

The Rodrigues rotation formula is a useful method for generating rotation matrices. Given a rotation axis and an angle, we can readily generate a rotation matrix using this method. $\mathbf{SO}(n)$ is a Lie group and the space of skew-symmetric matrices of dimension n is the Lie algebra of $\mathbf{SO}(n)$. We denote this Lie algebra by $\mathfrak{so}(n)$. The exponential map defined by the standard matrix exponential series e^A connects it to the Lie group. For any skew-symmetric matrix A, the matrix exponential e^A always gives a rotation matrix. This method is known as the Rodrigues formula.

3.1.1. Elliptical Rotations in the Plane. According to (3) a skew symmetric matrix can be expressed as

$$T = \begin{bmatrix} 0 & -\sqrt{a_2}/\sqrt{a_1} \\ \sqrt{a_1}/\sqrt{a_2} & 0 \end{bmatrix}.$$

The equality $T^t\Omega = -\Omega T$ is satisfied. The characteristic polynomial of T is, $P(x) = x^2 + 1$. So, $T^2 = -I$. We can obtain the elliptical rotation matrix easily using the matrix exponential (Fig. 1).

Theorem 1. *Let T be a $\mathcal{B}$-skew symmetric matrix. Then, the matrix exponential*

$$R^{\mathcal{B}}_{\theta} = e^{T\theta} = I + (\sin\theta)T + (1 - \cos\theta)T^2 = \begin{bmatrix} \cos\theta & -\dfrac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta \\ \dfrac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta & \cos\theta \end{bmatrix}$$

gives an elliptical rotation along the ellipse $a_1x^2 + a_2y^2 = \lambda$, $\lambda, a_1, a_2 \in \mathbb{R}^+$. That is, $R^{\mathcal{B}}_{\theta}$ is an elliptical rotation matrix in the space $\mathbb{R}^3_{a_1,a_2}$.

Remark 1. All similar ellipses have identical elliptical rotation matrices.

Example 1. Let's consider the ellipse $(\mathbf{E}_1) : \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ with the parametrization $\alpha(\theta) = (3\cos\theta, 2\sin\theta)$. Let's take the points

$$A = \alpha(\pi/4) = (3\sqrt{2}/2, \sqrt{2}) \quad \text{and}$$
$$B = \alpha(\pi/4 + \pi/3) = (3\sqrt{2}/4 - 3\sqrt{6}/4, \sqrt{2}/2 + \sqrt{6}/2).$$

That is, if we rotate the point A elliptically through the angle $\pi/3$, we get B. Now, let's find elliptical rotation matrix for these ellipses using the above theorem and get same results. To calculate the elliptical rotation matrix, first, we choose the elliptical inner product

$$\mathcal{B}(\mathbf{u}, \mathbf{w}) = \frac{u_1w_1}{9} + \frac{u_2w_2}{4}.$$

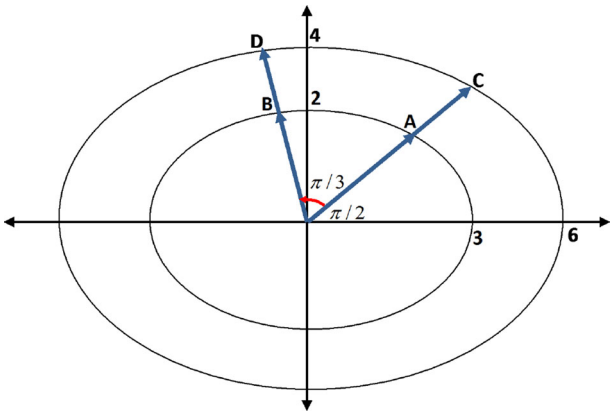


FIGURE 1. $\alpha(\theta) = (3\cos\theta, 2\sin\theta)$ and $\beta(\theta) = (6\cos\theta, 4\sin\theta)$

in accordance with the ellipse $(\mathbf{E}_1)$ such that $\mathbf{u} = (u_1, u_2)$ and $\mathbf{w} = (w_1, w_2)$. That is, our space is $\mathbb{R}^2_{1/9,1/4}$, $\Delta = 1/6$ and the $\mathcal{B}$-skew symmetric matrix has the form

$$T = \begin{bmatrix} 0 & -3/2 \\ 2/3 & 0 \end{bmatrix}.$$

Note that, $T^2 = -I$. We can obtain elliptical rotation matrix as,

$$R_\theta^{\mathcal{B}} = e^{\theta T} = \begin{bmatrix} \cos\theta & -\dfrac{3\sin\theta}{2} \\ \dfrac{2\sin\theta}{3} & \cos\theta \end{bmatrix}.$$

where $R_\theta^{\mathcal{B}}$ is a $\mathcal{B}$-orthogonal matrix in $\mathbb{R}^2_{1/9,1/4}$. Namely, the equalities $\det R_\theta^{\mathcal{B}} = 1$ and $(R_\theta^{\mathcal{B}})^t \Omega (R_\theta^{\mathcal{B}}) = \Omega$ are satisfied. For $\theta = \pi/3$, we get

$$R_{\pi/3}^{\mathcal{B}} = \begin{bmatrix} 1/2 & -3\sqrt{3}/4 \\ \sqrt{3}/3 & 1/2 \end{bmatrix}.$$

So, if we rotate the point A elliptically, we get $B = R_\theta^{\mathcal{B}}(A) = (3\sqrt{2}/4 - 3\sqrt{6}/4, \sqrt{2}/2 + \sqrt{6}/2)$. Thus, we get same result using elliptical rotation matrix for $(\mathbf{E}_1)$. Note that, $\|R_\theta^{\mathcal{B}}(A)\|_{\mathcal{B}} = 1$ and the angle between $\mathbf{x} = \overrightarrow{OA}$ and $\mathbf{y} = \overrightarrow{OB}$ is $\cos\theta = \frac{\mathcal{B}(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|_{\mathcal{B}}\|\mathbf{y}\|_{\mathcal{B}}} = \frac{1}{2}$. It can be seen that the elliptical rotation matrix $R_\theta^{\mathcal{B}}$ can be also used to interpret the motion on a similar ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$ to $(\mathbf{E}_1)$.

3.1.2. 3-Dimensional Elliptical Rotations. Let's take the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ where $a_i \in \mathbb{R}^+$. The scalar product for this ellipsoid is

$$\mathcal{B}(\mathbf{u}, \mathbf{w}) = a_1u_1w_1 + a_2u_2w_2 + a_3u_1w_3,$$

for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Also, the vector product is

$$\begin{aligned}\mathcal{V}(\mathbf{u} \times \mathbf{v}) &= \Delta \det \begin{bmatrix} \mathbf{e}_1/a_1 & \mathbf{e}_2/a_2 & \mathbf{e}_3/a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &= \Delta \begin{bmatrix} 0 & -u_3/a_1 & u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = T(\mathbf{v}^t)\end{aligned}$$

where $\Delta = \sqrt{a_1 a_2 a_3}$. The matrix

$$T = \Delta \begin{bmatrix} 0 & -u_3/a_1 & u_2/a_1 \\ u_3/a_2 & 0 & -u_1/a_2 \\ -u_2/a_3 & u_1/a_3 & 0 \end{bmatrix} \tag{6}$$

is skew symmetric in $\mathbb{R}^3_{a_1,a_2,a_3}$. That is, $T^t\Omega = -\Omega T$. So, the vector product in $\mathbb{R}^3_{a_1,a_2,a_3}$ can be viewed as a linear transformation, which corresponds to multiplication by a skew symmetric matrix. The characteristic polynomial of T is, $P(x) = x^3 + \|\mathbf{u}\|^2 x$ whose eigenvalues are $x_1 = 0$ and $x_{2,3} = \pm\|\mathbf{u}\| i$. According to characteristic polynomial $T^3 + \|\mathbf{u}\|^2 T = 0$. So, if we take a unit vector $\mathbf{u} \in \mathbb{R}^3_{a_1,a_2,a_3}$, we get $T^3 = -T$ and we can use Rodrigues and Cayley formulas.

Theorem 2. *Let T be a skew symmetric matrix in the form (6) such that $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3_{a_1,a_2,a_3}$ is a unit vector. Then, the matrix exponential*

$$R_\theta^{\mathcal{B},\mathbf{u}} = e^{T\theta} = I + (\sin\theta)T + (1 - \cos\theta)T^2$$

gives an elliptical rotation on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$, $a_i \in \mathbb{R}^+$. Furthermore, the matrix $R_\theta^{\mathcal{B},\mathbf{u}}$ can be expressed as

$$\begin{bmatrix} a_1u_1^2+(1-a_1u_1^2)\cos\theta & -\frac{\Delta u_3 \sin\theta}{a_1} - a_2u_1u_2(\cos\theta - 1) & \frac{\Delta u_2 \sin\theta}{a_1} - a_3u_1u_3(\cos\theta - 1) \\ \frac{\Delta u_3 \sin\theta}{a_2} - a_1u_1u_2(\cos\theta - 1) & a_2u_2^2+(1-a_2u_2^2)\cos\theta & -\frac{\Delta u_1 \sin\theta}{a_2} - a_3u_2u_3(\cos\theta - 1) \\ -\frac{\Delta u_2 \sin\theta}{a_3} - a_1u_1u_3(\cos\theta - 1) & \frac{\Delta u_1 \sin\theta}{a_3} - a_2u_2u_3(\cos\theta - 1) & a_3u_3^2+(1-a_3u_3^2)\cos\theta \end{bmatrix} \tag{7}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the rotation axis and θ is the elliptical rotation angle.

Proof. Since $\mathbf{u}$ is a unit vector in $\mathbb{R}^3_{a_1,a_2,a_3}$, we have $T^3 = -T$. So, we get

$$\begin{aligned} R_\theta^{\mathcal{B},\mathbf{u}} &= e^{\theta T} = I + \theta T + \frac{\theta^2 T^2}{2!} + \frac{-\theta^3 T}{3!} + \frac{-\theta^4 T^2}{4!} + \frac{\theta^5 T}{5!} + \frac{\theta^6 T^2}{6!} \cdots \\ &= I + T\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + T^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right) \\ &= I + T\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + T^2\left(1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)\right) \\ &= I + (\sin\theta)T + (1 - \cos\theta)T^2. \end{aligned}$$

If we expand this formula using $-a_2u_2^2 - a_3u_3^2 = a_1u_1^2 - 1$, we can obtain the rotation matrix as (7). □

Example 2. A parametrization of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{9}=1$ is

$$\alpha(\theta,\beta)=(2\cos\theta\cos\beta,2\cos\theta\sin\beta,3\sin\theta)$$

where $\theta\in[0,\pi)$ and $\beta\in[0,2\pi)$. Let's take the points

$$\begin{aligned}A&=\alpha(30^\circ,30^\circ)=(3/2,\sqrt{3}/2,3/2)\quad\text{and}\quad B=\alpha(120^\circ,30^\circ)\\&=(-\sqrt{3}/2,-1/2,3\sqrt{3}/2)\end{aligned}$$

on the ellipsoid. Let's find the rotation matrix which is rotate the point A to B elliptically. We have $a_1=a_2=1/4$ and $a_3=1/9$. So, $\Delta=1/12$. First, using the vector product of $\mathbf{x}=\overrightarrow{OA}$ and $\mathbf{y}=\overrightarrow{OB}$ in $\mathbb{R}^3_{1/4,1/4,1/9}$, we find the rotation axis $\mathbf{u}$.

$$\mathcal{V}(\mathbf{x}\times\mathbf{y})=\frac{1}{12}\begin{vmatrix}4i & 4j & 9k\\ 3/2 & \sqrt{3}/2 & 3/2\\ -\sqrt{3}/2 & -\frac{1}{2} & 3\sqrt{3}/2\end{vmatrix}=(1,-\sqrt{3},0).$$

Since $\mathcal{V}(\mathbf{x}\times\mathbf{y})$ is unit vector in $\mathbb{R}^3_{1/4,1/4,1/9}$, we get $\mathbf{u}=(1,-\sqrt{3},0)$. Thus, we obtain the elliptical rotation matrix

$$R^{\mathbf{u}}_{\theta}T(\theta)=\frac{1}{12}\begin{bmatrix}9\cos\theta+3 & 3\sqrt{3}(\cos\theta-1) & -4\sqrt{3}\sin\theta\\ 3\sqrt{3}(\cos\theta-1) & 3\cos\theta+9 & -4\sin\theta\\ 9\sqrt{3}\sin\theta & 9\sin\theta & 12\cos\theta\end{bmatrix}$$

by using (7). This matrix describes an elliptical rotation on a great ellipse such that it is intersection of the ellipsoid and the plane passing through the origin and $\mathcal{B}$-orthogonal to $\mathbf{u}$. It can be easily found that equation of the plane is $x=\sqrt{3}y$. So, $R^{\mathbf{u}}_{\theta}$ represents an elliptical rotation over the great ellipse is $y^2+\frac{1}{9}z^2=1$, $y=\sqrt{3}x$. Also, the elliptical rotation angle is $\pi/2$, since $\cos\theta=\mathcal{B}(\mathbf{x},\mathbf{y})=0$ (Fig. 2). Thus, we find

$$R^{\mathbf{u}}_{\pi/2}=\frac{1}{12}\begin{bmatrix}3 & -3\sqrt{3} & -4\sqrt{3}\\ -3\sqrt{3} & 9 & -4\\ 9\sqrt{3} & 9 & 0\end{bmatrix}. \tag{8}$$

The matrix (8), rotates the point A to the point B elliptically over the great ellipse $\frac{x^2}{2}+\frac{z^2}{9}=1$, $y=x$.

Remark 2. The eigenvalues of the matrix (7) are, $x_1=e^{i\theta}$, $x_2=e^{-i\theta}$ and $x_3=1$. Also, the eigenvector corresponding to 1 is the vector $\mathbf{u}$, the rotation axis of the motion.

Remark 3. Two ellipsoids on the form $a_1x^2+a_2y^2+a_3z^2=\lambda_1$ and $a_1x^2+a_2y^2+a_3z^2=\lambda_2$ are called similar where $\lambda_1,\lambda_2\in\mathbb{R}^+$. All similar ellipsoids have identical elliptical rotation matrices.

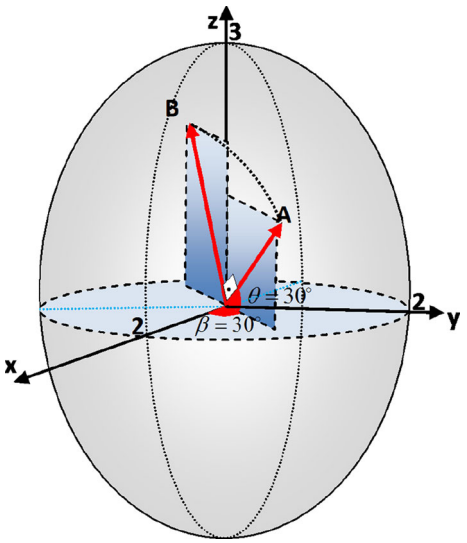


FIGURE 2. An example for elliptical rotation

3.2. Elliptical Cayley Rotation Matrix

In 1846, Arthur Cayley discovered a formula to express special orthogonal matrices using skew-symmetric matrices. It is called Cayley rotation matrix. In this section, we will describe the Cayley rotation matrix for any ellipsoid whose scalar product is $\mathcal{B}$. We call it $\mathcal{B}$-Cayley rotation matrix.

The Cayley rotation matrix is a useful tool that gives a parameterization for rotation matrices without the need to use trigonometric functions. Let A be an $n \times n$ skew symmetric matrix where $(I-A)$ is invertible. Cayley formula transforms the matrix A into $(I+A)(I-A)^{-1}$. $\mathbf{SO}(n)$ is isomorphic to $\mathfrak{so}(n)$ via the Cayley formula where $\mathfrak{so}(n)$ denotes the space of skew symmetric matrices, usually associated with the Lie algebra of the transformation group defined by rotations in $\mathbf{SO}(n)$. That is, the Cayley map is defined as

$$C : \mathfrak{so}(n) \to \mathbf{SO}(n)$$
$$A \to C(A) = (I + A)(I - A)^{-1}.$$

Note that, if a matrix M has the eigenvalue -1, then $(I - M)$ is not invertible. But, since A is skew symmetric, all its eigenvalues are purely imaginary in the Euclidean space and $(I - A)$ is invertible. That is, the Cayley formula is well-defined for all skew symmetric matrices. The inverse of the Cayley map is given by

$$C^{-1} = (I - A)(I + A)^{-1}.$$

In the Minkowski space, an eigenvalue of a skew symmetric matrix can be -1. In this case, the Cayley formula is not valid. For detailed information on the Cayley formula in Minkowski space see [3,7,21].

Theorem 4. *Let T be a skew symmetric matrix in the form (6) such that $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3_{a_1,a_2,a_3}$. Then,*

$$R^{\mathcal{B},\mathbf{u}} = (I+T)(I-T)^{-1}$$

is an elliptical rotation matrix on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ where $\mathbf{u}$ is the rotation axis. Furthermore, the matrix $R^{\mathcal{B},\mathbf{u}}$ can be written in the form

$$\frac{1}{1+\|\mathbf{u}\|^2_{\mathcal{B}}} \begin{bmatrix} a_1u_1^2 - u_2^2a_2 - u_3^2a_3 + 1 & a_2u_1u_2 - \frac{2\Delta u_3}{a_1} + u_1u_2a_2 & \frac{2\Delta u_2}{a_1} + a_3u_1u_3 + u_1u_3a_3 \\ \frac{2\Delta u_3}{a_2} + a_1u_1u_2 + u_1u_2a_1 & a_2u_2^2 - u_1^2a_1 - u_3^2a_3 + 1 & a_3u_2u_3 - \frac{2\Delta u_1}{a_2} + u_2u_3a_3 \\ a_1u_1u_3 - \frac{2\Delta u_2}{a_3} + u_1u_3a_1 & \frac{2\Delta u_1}{a_3} + a_2u_2u_3 + u_2u_3a_2 & a_3u_3^2 - u_1^2a_1 - u_2^2a_2 + 1 \end{bmatrix}. \tag{9}$$

Proof. Since, T is a $\mathcal{B}$-skew symmetric matrix, we have $T^t\Omega = -\Omega T$. Also, we can write

$$(I+T)^t\Omega = \Omega(I-T) \quad \text{and} \quad (I-T)^t\Omega = \Omega(I+T).$$

Using these equalities, it can be seen that

$$(R^{\mathcal{B},\mathbf{u}})^t\Omega(R^{\mathcal{B},\mathbf{u}}) = ((I+T)(I-T)^{-1})^t\Omega(I+T)(I-T)^{-1} = \Omega.$$

Also, since $\det(I+T) = 1 + \|\mathbf{u}\|_{\mathcal{B}}$ and $\det(I-T)^{-1} = \dfrac{1}{1+\|\mathbf{u}\|_{\mathcal{B}}}$, we have $\det R^{\mathcal{B},\mathbf{u}} = 1$. That is, $R^{\mathcal{B},\mathbf{u}}$ is an elliptical rotation matrix. The matrix (9) can be obtained after some tedious computations. □

Remark 4. The eigenvalues of the matrix (9) are

$$\lambda_1 = \frac{1-\|\mathbf{u}\|^2_{\mathcal{B}}}{1+\|\mathbf{u}\|^2_{\mathcal{B}}} + \frac{2\|\mathbf{u}\|_{\mathcal{B}}i}{1+\|\mathbf{u}\|^2_{\mathcal{B}}}, \quad \lambda_2 = \frac{1-\|\mathbf{u}\|^2_{\mathcal{B}}}{1+\|\mathbf{u}\|^2_{\mathcal{B}}} - \frac{2\|\mathbf{u}\|_{\mathcal{B}}i}{1+\|\mathbf{u}\|^2_{\mathcal{B}}} \quad \text{and} \quad \lambda_3 = 1.$$

Also, the eigenvector corresponding to 1 is $\mathbf{u}$, which is the rotation axis. The matrix (9) rotates a vector elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ about the axis $\mathbf{u} = (u_1, u_2, u_3)$ through the elliptical angle θ where

$$\tan\theta = \frac{2\|\mathbf{u}\|_{\mathcal{B}}}{1-\|\mathbf{u}\|^2_{\mathcal{B}}}. \tag{10}$$

Example 3. Let's find the elliptical rotation matrix representing a elliptical rotation on the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ about the axis $\mathbf{u} = (2, 3, 1)$. Using the matrix (9), we find

$$\begin{bmatrix} 0 & 0 & 2 \\ 3/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}.$$

The elliptical rotation angle corresponding to this matrix, can be found to be $-\pi/3$ from the formula (10).

3.3. Elliptical Householder Transformation

The Householder transformation was introduced in 1958 by Alston Scott Householder. A Householder transformation is a linear transformation in the form

$$\mathcal{H}_{\mathbf{v}}(x) = x - \frac{2\mathbf{v}\mathbf{v}^t}{\mathbf{v}^t\mathbf{v}}x$$

where $\mathbf{v}$ is a nonzero vector. This transformation describes a reflection about a plane or hyperplane passing through the origin and orthogonal to $\mathbf{v}$. Householder transformations on spaces with a non-degenerate bilinear or sesquilinear forms are studied in [15]. Every orthogonal transformation is the combination of reflections with respect to hyperplanes. This is known as the Cartan–Dieudonné theorem. A constructive proof of the Cartan–Dieudonné theorem for the case of generalized scalar product spaces is given by Uhlig [29] and Fuller [8]. An alternative proof of the Cartan–Dieudonné theorem for generalized real scalar product spaces is given by Rodríguez-Andrade and etc. [26]. They used the Clifford algebras to compute the factorization of a given orthogonal transformation as a product of reflections.

Householder transformations are widely used in tridiagonalization of symmetric matrices and to perform QR decompositions in numerical linear algebra [9]. Householder Transformation is cited in the top 10 algorithms of the twentieth century [5]. Generalized Householder matrices are the simplest generalized orthogonal matrices. A generalized Householder, or $\mathcal{B}$-Householder, matrix has the form

$$\mathcal{H}_{\mathbf{v}}(x) = x - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}x$$

for a non-B-isotropic vector $\mathbf{v}$ (i.e., $\mathbf{v}^t\Omega\mathbf{v} \neq 0$) [8,26].

In this paper, we use the elliptical version of the Householder transformation to generate an elliptical rotation matrix in a real elliptical scalar product space $\mathbb{R}^n_{a_1,a_2,\ldots,a_n}$. Let $\mathcal{H}_{\mathbf{v}} = [h_{ij}]_{n\times n}$ be a Householder matrix. Then

$$h_{ij} = \delta_{ij} - \frac{2v_iv_j\Omega_{jj}}{\mathbf{v}^t\Omega\mathbf{v}}$$

where δ_{ij} is the Kronecker delta. The Householder matrix $\mathcal{H}_{\mathbf{v}}$ is a $\mathcal{B}$-symmetric and $\mathcal{B}$-orthogonal matrix. To prove these, first we will show that $\mathcal{H}^t_{\mathbf{v}}\Omega = \Omega\mathcal{H}_{\mathbf{v}}$: $\mathbf{v}^t\Omega\mathbf{v} = \mathcal{B}(\mathbf{v},\mathbf{v}) = \|\mathbf{v}\|^2_{\mathcal{B}}$ is a positive real number. So,

$$\mathcal{H}^t_{\mathbf{v}}\Omega = \left(I - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\right)^t\Omega = \left(\Omega - \frac{2\Omega\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\right) = \Omega\left(I - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\right) = \Omega\mathcal{H}_{\mathbf{v}},$$

and $\mathcal{H}_{\mathbf{v}}$ is a $\mathcal{B}$-symmetric. Next, we will show that $\mathcal{H}^t_{\mathbf{v}}\Omega\mathcal{H}_{\mathbf{v}} = \Omega$.

$$\begin{aligned}
\mathcal{H}^t_{\mathbf{v}}\Omega\mathcal{H}_{\mathbf{v}} &= \Omega\mathcal{H}^2_{\mathbf{v}} \quad (\mathcal{H}_{\mathbf{v}} \text{ is } B\text{-symmetric})\\
&= \Omega\left(I - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\right)\left(I - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\right)\\
&= \Omega\left(I - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}} - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}} + \frac{4\,(\mathbf{v}\mathbf{v}^t\Omega)\,(\mathbf{v}\mathbf{v}^t\Omega)}{(\mathbf{v}^t\Omega\mathbf{v})^2}\right)\\
&= \Omega.
\end{aligned}$$

So, $\mathcal{H}_{\mathbf{v}}$ is $\mathcal{B}$-orthogonal. Using the above transformations, we can write the elliptical reflection matrix as

$$\mathcal{H}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|_{\mathcal{B}}^2} \begin{bmatrix} \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_1v_1^2 & -2a_2v_1v_2 & -2a_3v_1v_3 \\ -2a_1v_2v_1 & \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_2v_2^2 & -2a_3v_2v_3 \\ -2a_1v_3v_1 & -2a_2v_3v_2 & \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_3v_3^2 \end{bmatrix} \tag{11}$$

for the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$.

Corollary 1. *The Householder transformation doesn't change the length of a vector in $\mathbb{R}^n_{a_1,a_2,a_3}$. It describes an elliptical reflection about a plane passing through the origin $\mathcal{B}$-orthogonal to $\mathbf{v}$. Using the matrix* (11)*, we can express elliptical reflections on any ellipsoid.*

Example 4. Let's find the elliptical reflection matrix which reflects a point elliptically on the ellipsoid $2x^2+2y^2+z^2 = 1$, about the plane which is passes through the origin and is $\mathcal{B}$-orthogonal to $\mathbf{v} = (1, 2, 3)$. Our elliptical inner product is $\mathcal{B}(x, y) = 2x_1y_2+2x_2y_2+x_3y_3$. So, $\|\mathbf{v}\|_{\mathcal{B}}^2 = 2\cdot1^2+2\cdot2^2+1\cdot3^2 = 19$ and we get

$$\mathcal{H}_v = \begin{bmatrix} 15/19 & -8/19 & -6/19 \\ -8/19 & 3/19 & -12/19 \\ -12/19 & -24/19 & 1/19 \end{bmatrix}.$$

The determinant of this matrix is -1 and the equality $\mathcal{H}_{\mathbf{v}}^t\Omega\mathcal{H}_{\mathbf{v}} = \Omega$ holds. We find the reflection of $A(1/2, 1/2, 0)$ to be $B(7/38, -5/38, -18/19)$. Let's check it : The middle point of $[AB]$ is $C(13/38, 7/38, -9/19)$. The equation of the plane is $\mathcal{B}(\mathbf{v}, \mathbf{x}) = 0$ where $\mathbf{x} = (x, y, z)$. In our case, it is $2x+4y+3z = 0$, so C lies on this plane. Also, it can be easily seen that $\overrightarrow{AB}$ is $\mathcal{B}$-orthogonal to the plane.

Remark 5. For the n dimensional ellipsoid associated with the scalar product $\mathcal{B}$, the elliptical reflection matrix $\mathcal{H}_{\mathbf{v}}$ is in the form

$$\mathcal{H}_{\mathbf{v}}^{\mathcal{B}} = \frac{1}{\|\mathbf{v}\|_{\mathcal{B}}^2} \begin{bmatrix} \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_1v_1^2 & -2a_2v_1v_2 & -2a_3v_1v_3 & \cdots & -2a_nv_1v_n \\ -2a_1v_2v_1 & \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_2v_2^2 & -2a_3v_2v_3 & \cdots & -2a_nv_2v_n \\ -2a_1v_3v_1 & -2a_2v_3v_2 & \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_3v_3^2 & \cdots & -2a_nv_3v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2a_1v_nv_1 & -2a_2v_nv_2 & -2v_na_3v_3 & \cdots & \|\mathbf{v}\|_{\mathcal{B}}^2 - 2a_nv_n^2 \end{bmatrix}$$

where $\mathbf{v}^t = (v_1, v_2, \ldots, v_n)$.

Using the following theorems, for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of same length, we can determine the elliptical Householder transformations $\mathcal{H}_{\mathbf{x}-\mathbf{y}}^{\mathcal{B}}$ and $\mathcal{H}_{\mathbf{x}+\mathbf{y}}^{\mathcal{B}}$ that reflect $\boldsymbol{x}$ to $\boldsymbol{y}$ and $\boldsymbol{x}$ to $-\boldsymbol{y}$ respectively. The composition of these two transformations give us the elliptical rotation matrix that elliptically rotates $\boldsymbol{x}$ to $\boldsymbol{y}$.

Theorem 5. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_{\mathcal{B}}$ be any two nonzero vectors where $\|\boldsymbol{x}\|_{\mathcal{B}} = \|\boldsymbol{y}\|_{\mathcal{B}}$. If $\mathbf{v} = \boldsymbol{x} - \boldsymbol{y}$, then*

$$\mathcal{H}^{\mathcal{B}}_{\mathbf{v}}(\mathbf{x}) = \boldsymbol{x} - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\boldsymbol{x} = \boldsymbol{y}.$$

Proof. By direct computation, we find

$$\mathbf{v}^t\Omega\mathbf{v} = \mathcal{B}(\boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}) = 2\|\boldsymbol{x}\|^2_{\mathcal{B}} - 2\boldsymbol{y}^t\Omega\boldsymbol{x}$$

and

$$2\mathbf{v}\mathbf{v}^t\Omega\boldsymbol{x} = 2(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^t\Omega\boldsymbol{x} = (\boldsymbol{x}-\boldsymbol{y})(2\|\boldsymbol{x}\|^2_{\mathcal{B}} - 2\boldsymbol{y}^t\Omega\boldsymbol{x}).$$

Using these equalities, we obtain

$$\mathcal{H}^{\mathcal{B}}_{\mathbf{v}}(\mathbf{x}) = \boldsymbol{x} - \frac{2\mathbf{v}\mathbf{v}^t\Omega}{\mathbf{v}^t\Omega\mathbf{v}}\,\boldsymbol{x} = \boldsymbol{x} - \frac{(\boldsymbol{x}-\boldsymbol{y})(2\|\boldsymbol{x}\|^2_{\mathcal{B}} - 2\boldsymbol{y}^t\Omega\boldsymbol{x})}{2\|\boldsymbol{x}\|^2_{\mathcal{B}} - 2\boldsymbol{y}^t\Omega\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y}.$$

□

Theorem 6. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_{\mathcal{B}}$ be any two nonzero vectors where $\|\boldsymbol{x}\|_{\mathcal{B}} = \|\boldsymbol{y}\|_{\mathcal{B}}$. If $\mathbf{v} = \boldsymbol{x} + \boldsymbol{y}$, then*

$$\mathcal{H}^{\mathcal{B}}_{\mathbf{v}}(\boldsymbol{x}) = -\boldsymbol{y}.$$

Proof. By direct computation, we see that

$$\mathbf{v}^t\Omega\mathbf{v} = \mathcal{B}(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}) = 2\|\boldsymbol{x}\|^2_{\mathcal{B}} + 2\boldsymbol{y}^t\Omega\boldsymbol{x}$$

and

$$2\mathbf{v}\mathbf{v}^t\Omega\boldsymbol{x} = 2(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}+\boldsymbol{y})^t\Omega\boldsymbol{x} = (\boldsymbol{x}+\boldsymbol{y})(2\|\boldsymbol{x}\|^2_{\mathcal{B}} + 2\boldsymbol{y}^t\Omega\boldsymbol{x}).$$

Thus, we get

$$\mathcal{H}^{\mathcal{B}}_{\mathbf{v}}(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{x}+\boldsymbol{y}) = -\boldsymbol{y}\,.$$

Moreover, we have $\mathcal{H}^{\mathcal{B}}_{\boldsymbol{y}}(-\boldsymbol{y}) = \boldsymbol{y}$.

$$\mathcal{H}^{\mathcal{B}}_{\mathbf{y}}(-\boldsymbol{y}) = -\boldsymbol{y} + \frac{2\boldsymbol{y}\boldsymbol{y}^t\Omega}{\boldsymbol{y}^t\Omega\boldsymbol{y}}\boldsymbol{y} = -\boldsymbol{y} + \frac{2\boldsymbol{y}(\boldsymbol{y}^t\Omega\boldsymbol{y})}{\boldsymbol{y}^t\Omega\boldsymbol{y}} = \boldsymbol{y}.$$

which means that $\mathcal{H}^{\mathcal{B}}_{\boldsymbol{y}}\mathcal{H}^{\mathcal{B}}_{\boldsymbol{x}+\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{y}$. Hence, we obtain the elliptical rotation matrix

$$R_{\mathcal{B}} = \mathcal{H}^{\mathcal{B}}_{\boldsymbol{y}}\mathcal{H}^{\mathcal{B}}_{\boldsymbol{x}+\boldsymbol{y}}$$

which rotates $\mathbf{x}$ to $\mathbf{y}$ elliptically. The rotation axis is the vector $\mathcal{V}(\mathbf{x}\times\mathbf{y})$ and the elliptic rotation angle can be found using elliptical inner product. □

Example 5. Let's find the elliptic rotation matrix that rotates $\boldsymbol{x} = (0,0,1)$ to $\boldsymbol{y} = (2/5, 2/5, 3/5)$ on the ellipsoid $2x^2 + 2y^2 + z^2 = 1$. Using the matrix (11), we get

$$\mathcal{H}_{\boldsymbol{x}+\boldsymbol{y}} = \frac{1}{5}\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & -2 \\ -4 & -4 & -3 \end{bmatrix} \quad \text{and} \quad \mathcal{H}_{\boldsymbol{y}} = \frac{1}{25}\begin{bmatrix} 9 & -16 & -12 \\ -16 & 9 & -12 \\ -24 & -24 & 7 \end{bmatrix}.$$

Thus, the rotation matrix is

$$R = \mathcal{H}_{\boldsymbol{y}}\mathcal{H}_{\boldsymbol{x}+\boldsymbol{y}} = \frac{1}{5}\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ -4 & -4 & 3 \end{bmatrix}$$

which satisfies $R(\boldsymbol{x}) = \boldsymbol{y}$, $\det R = 1$, and $R^t\Omega R = \Omega$ where $\Omega = diag(2, 2, 1)$. Note that, we can obtain same matrix using (7).

3.4. Elliptic Quaternions

Quaternions were discovered by Sir William R. Hamilton in 1843 and the theory of quaternions was expanded to include applications such as rotations in the early twentieth century. The most important property of the quaternions is that every unit quaternion represents a rotation and this plays an important role in the study of rotations in 3-dimensional vector spaces. Using unit quaternions is a useful, natural, and elegant way to perceive rotations. Quaternions are used especially in computer vision, computer graphics, animation, and kinematics.

Quaternion algebra $\mathbb{H}$ is an associative, non-commutative division ring with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. We can express any quaternion q as $q = (q_1, q_2, q_3, q_4) = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ or $q = S_q + \mathbf{V}q$ where the symbols $S_q = q_1$ and $\mathbf{V}_q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ denote the scalar and vector parts of q, respectively. If $S_q = 0$ then q is called a pure quaternion. The conjugate of q is denoted by $\overline{q}$, and defined as $\overline{q} = S_q - \mathbf{V}_q$. The norm of a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined by $\sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and is denoted by N_q. We say that $q_0 = q/N_q$ is unit quaternion if $q \neq 0$. The set of unit quaternions are denoted by $\mathbb{H}_1$. Every unit quaternion can be written in the form $q_0 = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ where $\boldsymbol{\varepsilon}_0$ is a unit vector satisfying the equality $\boldsymbol{\varepsilon}_0^2 = -1$. It is called the axis of the quaternion [27,30,31].

Let q and r be two quaternions. Then, the linear transformation $R_q : \mathbb{H} \to \mathbb{H}$ defined by $R_q(r) = qrq^{-1}$ is a quaternion that has the same norm and scalar as r. Since the scalar part of the quaternion r doesn't change under R_q, we will only examine how its vector part $\mathbf{V}_r$ changes under the transformation R_q. We can interpret the rotation of a vector in the Euclidean 3-space using the quaternion product $q\mathbf{V}_r q^{-1}$.

If $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ is a unit quaternion, then, using the linear transformation $R_q(\mathbf{V}_r) = q\mathbf{V}_r q^{-1}$, the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_1q_2 + 2q_3q_0 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_1q_0 \\ 2q_1q_3 - 2q_2q_0 & 2q_1q_0 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \quad (4)$$

This rotation matrix represents a rotation through angle 2θ about the axis $\boldsymbol{\varepsilon} = (q_1, q_2, q_3)$. In the Lorentzian space, the rotation matrix corresponding to a timelike quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is,

$$R_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix}$$

where the set of unit timelike quaternions satisfies the properties $q\overline{q} = 1$, $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1$ (see [19]).

3.4.1. Elliptic Quaternions. To get an elliptical rotation matrix, firs we define the set of elliptic quaternions suitable for the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$. Let's take four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities

$$\mathbf{i}^2 = -a_1, \quad \mathbf{j}^2 = -a_2, \quad \mathbf{k}^2 = -a_3$$

and

$$\mathbf{ij} = \frac{\Delta}{a_3}\mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \frac{\Delta}{a_1}\mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \frac{\Delta}{a_2}\mathbf{j} = -\mathbf{ik}$$

where $a_1, a_2, a_3 \in \mathbb{R}^+$ and $\Delta = \sqrt{a_1a_2a_3}$.

The set of elliptic quaternions will be denoted by $\mathbb{H}_{a_1,a_2,a_3}$. This set is an associative, non-commutative division ring with our basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. If we take $a_1 = a_2 = a_3 = 1$, we get the usual quaternion algebra. The elliptic quaternion product table is given below.

	1	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	$-a_1$	$\Delta\mathbf{k}/a_3$	$-\Delta\mathbf{j}/a_2$
j	**j**	$-\Delta\mathbf{k}/a_3$	$-a_2$	$\Delta\mathbf{i}/a_1$
k	**k**	$\Delta\mathbf{j}/a_2$	$-\Delta\mathbf{i}/a_1$	$-a_3$

The elliptic quaternion product of two quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as

$$p_0q_0 - \mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) + p_0\mathbf{V}_q + q_0\mathbf{V}_p + \mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) \tag{12}$$

where $\mathcal{B}(\mathbf{V}_p, \mathbf{V}_q)$ and $\mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q)$ are the elliptical scalar product and the elliptical vector product, respectively. If p and q are pure, then

$$\begin{aligned} pq &= -\mathcal{B}(\mathbf{V}_p, \mathbf{V}_q) + \mathcal{V}(\mathbf{V}_p \times \mathbf{V}_q) \\ &= -(a_1p_1q_1 + a_2p_2q_2 + a_3p_3q_3) + \Delta \begin{vmatrix} i/a_1 & j/a_2 & k/a_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}. \end{aligned}$$

The elliptic quaternion product for $\mathbb{H}_{a_1,a_2,a_3}$ can be expressed as

$$pq = \begin{bmatrix} p_0 & -a_1p_1 & -a_2p_2 & -a_3p_3 \\ p_1 & p_0 & -\dfrac{p_3\Delta}{a_1} & \dfrac{p_2\Delta}{a_1} \\ p_2 & \dfrac{p_3\Delta}{a_2} & p_0 & -\dfrac{p_1\Delta}{a_3} \\ p_3 & -\dfrac{p_2\Delta}{a_3} & \dfrac{p_1\Delta}{a_3} & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

For example, let $p, q \in \mathbb{H}_{2,2,1}$. Then, the elliptic quaternion product of p and q defined is

$$pq = \begin{bmatrix} p_0 & -2p_1 & -2p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -2p_2 & 2p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

For $p = 1+2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $q = 2+4\mathbf{i}+\mathbf{j}+3\mathbf{k}$, we get $pq = (-32, 13, 17, -9)$. This can also be calculated using the product table

	1	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	-2	$2\mathbf{k}$	$-\mathbf{j}$
j	**j**	$-2\mathbf{k}$	-2	**i**
k	**k**	**j**	$-\mathbf{i}$	-1

.

Remember that the algebra is formed by a vector space $\mathbb{V}$ equipped with a quadratic form Q with the following equalities

$$\mathbf{v}^2 = Q(\mathbf{v});$$
$$\mathbf{uv} + \mathbf{vu} = 2\mathcal{B}_Q(\mathbf{u}, \mathbf{v})$$

is called a *Clifford algebra* and is denoted by $C\ell(\mathbb{V}, Q)$. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a base for an n-dimensional vector space $\mathbb{V}$, then $C\ell(\mathbb{V}, Q)$ is formed by the multivectors

$$\{1\} \cup \{\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k} : 1 \le i_1 \le \cdots \le i_k \le n,\ 1 \le k \le n\}$$

with $\dim(C\ell(\mathbb{V}, Q)) = 2^n$. Since the Clifford product of two even multivectors is an even multivector, they define an even subalgebra of $C\ell(\mathbb{V}, Q)$. The even subalgebra of an n-dimensional Clifford algebra is isomorphic to a Clifford algebra of $(n-1)$ dimensions and it is denoted by $C\ell^+(\mathbb{V}, Q)$. The Hamiltonian quaternion algebra $\mathbb{H}$ is isomorphic with the even subalgebra $C\ell^+_{3,0} = C\ell(\mathbb{R}^3, Q = x_1^2 + x_2^2 + x_3^2)$ by $\{1, e_2e_3 \to \mathbf{j}, e_1e_3 \to \mathbf{k}, e_1e_2 \to \mathbf{i}\}$ and the split quaternion algebra $\widehat{\mathbb{H}}$ is isomorphic with the even subalgebra $C\ell^+_{2,1} = C\ell(\mathbb{R}^3_1, Q = -x_1^2 + x_2^2 + x_3^2)$ by $\{1, e_2e_3 \to \mathbf{i}, e_3e_1 \to \mathbf{k}, e_1e_2 \to \mathbf{j}\}$ [27]. Similarly, the elliptic quaternion algebra is an even subalgebra of the Clifford algebra

$$C\ell(\mathbb{R}^3) = \big\{q = q_0 + \mathbf{e}_1q_1 + \mathbf{e}_2q_2 + \mathbf{e}_3q_3 : \mathbf{e}_1^2 = a_1,\ \mathbf{e}_2^2 = a_2,$$
$$\mathbf{e}_3^2 = a_3,\ \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0\big\}$$

associated with the nondegenerate quadratic form $Q(x) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2$ and is denoted by $C\ell^+(\mathbb{R}^3, a_1x_1^2 + a_2x_2^2 + a_3x_3^2)$, or shortly $C\ell^+(\mathbb{R}^3_{a_1,a_2,a_3})$. $\mathbb{H}_{a_1,a_2,a_3}$ is isomorphic to $C\ell^+(\mathbb{R}^3_{a_1,a_2,a_3})$ with $\{1, \frac{a_1}{\Delta}e_2e_3 \to \mathbf{i}, \frac{a_2}{\Delta}e_1e_3 \to \mathbf{j}, \frac{a_3}{\Delta}e_1e_2 \to \mathbf{k}\}$.

For the quadratic form $Q = a_1x_1^2 + a_1x_2^2 + a_1x_3^2$, recall that the elliptical inner product can be obtained by using the equality

$$\mathcal{B}_Q(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})\right].$$

So, we get $\mathcal{B}_Q(\mathbf{x}, \mathbf{y}) = a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3$ for $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Thus, we can construct an elliptic quaternion algebra for any elliptical inner product space.

Conjugate, norm and inverse of an elliptic quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be defined similar to usual quaternions:

$$\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}, \qquad N_q = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + a_1q_1^2 + a_2q_2^2 + a_3q_3^2},$$
$$q^{-1} = \frac{\overline{q}}{N_q}.$$

Also, each elliptic quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be written in the form

$$q_0 = N_q(\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta)$$

where

$$\cos\theta = \frac{q_0}{N_q} \quad \text{and} \quad \sin\theta = \frac{\sqrt{a_1q_1^2 + a_2q_2^2 + a_3q_3^2}}{N_q}.$$

Here, $\boldsymbol{\varepsilon}_0 = \dfrac{(q_1, q_2, q_3)}{\sqrt{a_1q_1^2 + a_2q_2^2 + a_3q_3^2}}$ is a unit vector in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$ satisfying the equality $\boldsymbol{\varepsilon}_0^2 = -1$. It is called the axis of the rotation. For example, if $q = 1 + 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} \in \mathbb{H}_{2,2,1}$, then $N_q = \sqrt{1^2 + 2\cdot 2^2 + 2\cdot 1^2 + 1\cdot 5^2} = 6$ and we can write

$$q = \frac{1}{6} + \frac{\sqrt{35}}{6}\frac{(2,1,5)}{\sqrt{35}} = \cos\theta + \frac{(2,1,5)}{\sqrt{35}}\sin\theta$$

where $\boldsymbol{\varepsilon}_0 = \dfrac{1}{\sqrt{35}}(2,1,5)$ is a unit vector in $\mathbb{R}^3_{2,2,1}$ with $\boldsymbol{\varepsilon}_0^2 = -1$.

Theorem 7. *Each a unit elliptic quaternion represents an elliptical rotation on the an ellipsoid. If*

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta \in \mathbb{H}_{a_1,a_2,a_3}$$

is a unit elliptic quaternion, then the linear map $R_\theta(\mathbf{v}) = q\mathbf{v}q^{-1}$ gives an elliptical rotation through the elliptical angle 2θ, about the axis $\boldsymbol{\varepsilon}_0$, where $\mathbf{v} \in \mathbb{R}^3$. The elliptical rotation matrix to corresponding to the quaternion q is

$$R_\theta^q = \begin{bmatrix} q_0^2 + a_1q_1^2 - a_2q_2^2 - a_3q_3^2 & 2a_2q_1q_2 - 2\dfrac{q_0q_3\Delta}{a_1} & 2a_3q_1q_3 + 2\dfrac{q_0q_2\Delta}{a_1} \\ 2a_1q_1q_2 + 2\dfrac{q_0q_3\Delta}{a_2} & q_0^2 - a_1q_1^2 + a_2q_2^2 - a_3q_3^2 & 2a_3q_2q_3 - 2\dfrac{q_0q_1\Delta}{a_2} \\ 2a_1q_1q_3 - 2\dfrac{q_0q_2\Delta}{a_3} & 2a_2q_2q_3 + 2\dfrac{q_0q_1\Delta}{a_3} & q_0^2 - a_1q_1^2 - a_2q_2^2 + a_3q_3^2 \end{bmatrix} \tag{13}$$

for the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$.

Proof. It can be seen that R_θ is a linear transformation and preserves the norm. Using the equalities,

$$\begin{aligned}
R_\theta(\mathbf{i}) &= (a_1q_1^2q_0^2 - q_2^2a_2 - q_3^2a_3)\mathbf{i} + 2\left(a_1q_1q_2 + q_0q_3\sqrt{\frac{a_1a_3}{a_2}}\right)\mathbf{j} \\
&\quad +2\left(q_1q_3a_1 - q_0q_2\sqrt{\frac{a_1a_2}{a_3}}\right)\mathbf{k}, \\
R_\theta(\mathbf{j}) &= 2\left(a_2q_1q_2 - q_0q_3\sqrt{\frac{a_2a_3}{a_1}}\right)\mathbf{i} + (a_2q_2^2 + q_0^2 - q_1^2a_1 - q_3^2a_3)\mathbf{j} \\
&\quad +2\left(a_2q_2q_3 + q_0q_1\sqrt{\frac{a_1a_2}{a_3}}\right)\mathbf{k}, \\
R_\theta(\mathbf{k}) &= 2\left(a_3q_1q_3 + q_0q_2\sqrt{\frac{a_2a_3}{a_1}}\right)\mathbf{i} + 2\left(a_3q_2q_3 - q_0q_1\sqrt{\frac{a_1a_3}{a_2}}\right)\mathbf{j} \\
&\quad +(a_3q_3^2 + q_0^2 - q_1^2a_1 - q_2^2a_2)\mathbf{k},
\end{aligned}$$

we can obtain (13). So, the rotation matrix (13) is an elliptical rotation matrix on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$. That is, the equalities $\det R_\theta = 1$ and $R_\theta^t \Omega R_\theta = \Omega$ are satisfied. Also, note that, if we take $a_1 = a_2 = a_3 = 1$, the standard rotation matrix is obtained. Now, let's choose an orthonormal set $\{\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ satisfying the equalities

$$\mathcal{V}(\boldsymbol{\varepsilon}_0 \times \boldsymbol{\varepsilon}_1) = \boldsymbol{\varepsilon}_2, \quad \mathcal{V}(\boldsymbol{\varepsilon}_2 \times \boldsymbol{\varepsilon}_0) = \boldsymbol{\varepsilon}_1, \quad \mathcal{V}(\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2) = \boldsymbol{\varepsilon}_0.$$

If $\boldsymbol{\varepsilon}$ is a vector in the plane of the $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_1$, we can write it as

$$\boldsymbol{\varepsilon} = \cos\alpha\boldsymbol{\varepsilon}_0 + \sin\alpha\boldsymbol{\varepsilon}_1.$$

To compute $R_\theta^q(\boldsymbol{\varepsilon}) = q\boldsymbol{\varepsilon}q^{-1}$, let's find how $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_1$ change under the transformation R_θ^q. Since $\mathbf{V}_q$ is parallel to $\boldsymbol{\varepsilon}_0$, we have $q\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0 q$ by (4) and $R_q(\boldsymbol{\varepsilon}_0) = q\boldsymbol{\varepsilon}_0q^{-1} = \boldsymbol{\varepsilon}_0qq^{-1} = \boldsymbol{\varepsilon}_0$. So, $\boldsymbol{\varepsilon}_0$ is not changed under the transformation R_θ^q. It means that $\boldsymbol{\varepsilon}_0$ is the rotation axis. On the other hand,

$$\begin{aligned}
R_q(\boldsymbol{\varepsilon}_1) &= q\boldsymbol{\varepsilon}_1q^{-1} \\
&= (\cos\theta + \boldsymbol{\varepsilon}_0\sin\theta)\boldsymbol{\varepsilon}_1(\cos\theta - \boldsymbol{\varepsilon}_0\sin\theta) \\
&= \boldsymbol{\varepsilon}_1\cos^2\theta - \cos\theta\sin\theta(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0) + \cos\theta\sin\theta(\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1) - (\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_0\sin^2\theta.
\end{aligned}$$

Since we know that $\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0 = \mathcal{V}(\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_0) = -\mathcal{V}(\boldsymbol{\varepsilon}_0 \times \boldsymbol{\varepsilon}_1) = -\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1 = -\boldsymbol{\varepsilon}_2$ for orthogonal, pure quaternions, we obtain

$$\begin{aligned}
R_q(\boldsymbol{\varepsilon}_1) &= \boldsymbol{\varepsilon}_1\cos^2\theta + (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0)\boldsymbol{\varepsilon}_0\sin^2\theta + 2\boldsymbol{\varepsilon}_2\cos\theta\sin\theta \\
&= \boldsymbol{\varepsilon}_1\cos^2\theta + \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0^2\sin^2\theta + 2\boldsymbol{\varepsilon}_2\cos\theta\sin\theta \\
&= \boldsymbol{\varepsilon}_1\cos 2\theta + \boldsymbol{\varepsilon}_2\sin 2\theta
\end{aligned}$$

That is, $\boldsymbol{\varepsilon}$ is rotated through the elliptical angle 2θ about $\boldsymbol{\varepsilon}_0$ by the transformation $R_q(\boldsymbol{\varepsilon})$. □

Corollary 1. *All elliptical rotations on an ellipsoid can be represented by elliptic quaternions which is defined for that ellipsoid.*

The matrix (13) is only rotation matrix in the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$ and shows rotations along an ellipse on the $a_1x^2+a_2y^2+a_3z^2=\lambda$, $\lambda\in\mathbb{R}^+$. Here, notice that the ellipsoid $a_1x^2+a_2y^2+a_3z^2=1$ is unit sphere of the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$. So, R^q_θ rotates a vector elliptically on the ellipsoid $a_1x^2+a_2y^2+a_3z^2=1$ or similar to this ellipsoid. Moreover, the matrix R^q_θ depends to elliptical inner product and elliptical vector product and we can write as $R^q_\theta\in\mathbf{SO}_{\mathcal{B}}(n)$ or $R^q_\theta\in\mathbf{SO}_{a_1,a_2,a_3}(n)$ where $q\in\mathbb{H}_{a_1,a_2,a_3}$. We must always use proper elliptic quaternion set and product corresponding to a predetermined ellipsoid.

Example 6. Let's find the general elliptical rotation matrix for the ellipsoid $2x^2+2y^2+z^2=1$. Using (13), we obtain,

$$R^q_\theta=\begin{bmatrix} q_0^2+2q_1^2-2q_2^2-q_3^2 & 4q_1q_2-2q_0q_3 & 2q_0q_2+2q_1q_3\\ 2q_0q_3+4q_1q_2 & q_0^2-2q_1^2+2q_2^2-q_3^2 & 2q_2q_3-2q_0q_1\\ 4q_1q_3-4q_0q_2 & 4q_0q_1+4q_2q_3 & q_0^2-2q_1^2-2q_2^2+q_3^2\end{bmatrix}.$$

Here, $\det R_\theta=(q_0^2+2q_1^2+2q_2^2+q_3^2)^3=1$ and $R^t_\theta\Omega R=\Omega$ where $\Omega=diag(2,2,1)$. For example, the unit quaternion $q=(0,1/2,1/2,0)$ represents an elliptical rotation on the ellipsoid $2x^2+2y^2+z^2=1$ through the elliptical angle π, about the axis $(1/2,1/2,0)$. And the elliptical rotation matrix is

$$R^q_\pi=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}.$$

Let p and q be two unit elliptic quaternions of the same kind. That is, let $p,q\in\mathbb{H}_{a_1,a_2,a_3}$. In this case, $R^p_{\theta_1}$ and $R^q_{\theta_2}$ are two elliptical rotation matrices on the scalar product space $\mathbb{R}^3_{a_1,a_2,a_3}$. That is $R^p_{\theta_1}$ and $R^q_{\theta_2}$ rotate a vector elliptically on the ellipsoid $a_1x^2+a_2y^2+a_3z^2=1$**.** The composition of these rotations can be expressed by the elliptic quaternion product qp. The axis and the elliptical angle of the composite rotation is given by the product qp. Let $R^p_{\theta_1}(\mathbf{u})=\mathbf{v}$ and $R^q_{\theta_2}(\mathbf{v})=\mathbf{w}$. Then

$$\begin{aligned}\mathbf{w}&=R^q_{\theta_2}(\mathbf{v})=R^q_{\theta_2}(R^p_{\theta_1}(\mathbf{u}))=qR^p_{\theta_1}(\mathbf{u})q^{-1}\\&=qp\mathbf{u}p^{-1}q^{-1}=(qp)\mathbf{u}(qp)^{-1}=R^{qp}_{\theta_3}(\mathbf{u}).\end{aligned}$$

It means that $R^q_{\theta_2}R^p_{\theta_1}=R^{qp}_{\theta_3}$. As an example, let's take the unit elliptic quaternions $q=(0,\frac{1}{2},\frac{1}{2},0)$ and $p=(\frac{1}{6},\frac{2}{6},\frac{1}{6},\frac{5}{6})$. That is, $p,q\in\mathbb{H}_{2,2,1}$. The scalar and the vector products are

$$\begin{aligned}\mathcal{B}(\mathbf{x},\mathbf{y})&=a_1x_1x_2+a_2y_1y_2+a_3z_1z_2,\\ \mathcal{V}(\mathbf{x}\times\mathbf{y})&=\left(\frac{\Delta}{a_1}(y_1z_2-y_2z_1),\frac{\Delta}{a_2}(-x_1z_2+x_2z_1),\frac{\Delta}{a_3}(x_1y_2-x_2y_1)\right)\end{aligned}$$

respectively. The elliptic quaternion product of the quaternions $p=p_0+p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k}$ and $q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$ is defined as

$$pq=p_0q_0-\mathcal{B}(\mathbf{V}_p,\mathbf{V}_q)+p_0\mathbf{V}_q+q_0\mathbf{V}_p+\mathcal{V}(\mathbf{V}_p\times\mathbf{V}_q).$$

So, qp can be found to be $qp=(-\frac{1}{2},\frac{1}{2},-\frac{1}{3},-\frac{1}{6})\in\mathbb{H}_{2,2,1}$.

Using (13), we can find as

$$R^q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R^p = \begin{bmatrix} -1/2 & -1/18 & 11/18 \\ 1/2 & -5/6 & 1/6 \\ 1 & 7/9 & 4/9 \end{bmatrix} \quad \text{and}$$

$$R^{qp} = \begin{bmatrix} 1/2 & -5/6 & 1/6 \\ -1/2 & -1/18 & 11/18 \\ -1 & -7/9 & -4/9 \end{bmatrix}.$$

It can be seen that the matrix equality $R^q R^p = R^{qp}$ is satisfied.

Remark 6. If p and q are two unit elliptic quaternions of different kinds, namely $p \in \mathbb{H}_{a_1,a_2,a_3}$ and $q \in \mathbb{H}_{b_1,b_2,b_3}$, then the composition of corresponding rotations cannot be expressed by the elliptic quaternion product. Because, for each quaternion, the scalar product space is different, hence the elliptic quaternion product is also different. $R^p_{\theta_1}$ and $R^q_{\theta_2}$ represent a rotations on the ellipsoids $a_1x^2 + a_2y^2 + a_3z^2 = 1$ and $b_1x^2 + b_2y^2 + b_3z^2 = 1$ respectively and the composition of the elliptical rotations on two different ellipsoids cannot be expressed by the elliptical quaternion product of two elliptical quaternions of different kinds.

3.5. An Algorithm

Generating 3-dimensional rotation matrix that rotates $\boldsymbol{x} = (x_1, y_1, y_1)$ to $\boldsymbol{y} = (x_2, y_2, z_2)$ elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$.

Step 1. Write, $\boldsymbol{a} = (a_1, a_2, a_3)$ for the given ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ where $a_1, a_2, a_3 \in \mathbb{R}^+$.
Step 2. Define Scalar Product $\mathcal{B}$, norm of a vector and Scalar product constant Δ as follows:

$$\begin{aligned} \mathcal{B}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}) &= \mathbf{B}(x_1, y_1, z_1, x_2, y_2, z_2, a_1, a_2, a_3) = a_1x_1x_2 + a_2y_1y_2 + a_3z_1z_2, \\ \mathcal{N}(\boldsymbol{x}, \boldsymbol{a}) &= \sqrt{\mathcal{B}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{a})}, \\ \Delta &= \sqrt{a_1a_2a_3} \end{aligned}$$

where $\boldsymbol{x} = (x_1, y_1, y_1)$ and $\boldsymbol{y} = (x_2, y_2, z_2)$.
Step 3. Define Vector Product $\mathcal{V}$ as

$$\begin{aligned} \mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}) &= \mathbf{V}(x_1, y_1, z_1, x_2, y_2, z_2, a_1, a_2, a_3) \\ &= \left(\frac{\Delta}{a_1}(y_1z_2 - y_2z_1), \frac{\Delta}{a_2}(-x_1z_2 + x_2z_1), \frac{\Delta}{a_3}(x_1y_2 - x_2y_1) \right). \end{aligned}$$

Step 4. Choose the vectors $\boldsymbol{x} = (x_1, y_1, y_1)$ and $\boldsymbol{y} = (x_2, y_2, z_2)$ to find the elliptical rotation matrix that rotates $\boldsymbol{x}$ to $\boldsymbol{y}$ elliptically on the ellipsoid.
Step 5. Find, $\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ and norm of the vectors $\boldsymbol{x}, \boldsymbol{y}$ and $\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$. That is, find $\mathcal{N}(\boldsymbol{x}, \boldsymbol{a})$, $\mathcal{N}(\boldsymbol{y}, \boldsymbol{a})$ and $\mathcal{N}(\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}), \boldsymbol{a})$.
Step 6. Find the rotation axis $\mathbf{u} = (u_1, u_2, u_3)$ where

$$\begin{aligned} u_1 &= \frac{\Delta(y_1z_2 - y_2z_1)}{a_1 \cdot \mathcal{N}(\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}), \boldsymbol{a})}, \; u_2 = \frac{-\Delta(x_1z_2 + x_2z_1)}{a_2 \cdot \mathcal{N}(\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}), \boldsymbol{a})}, \; u_1 \\ &= \frac{\Delta(x_1y_2 - x_2y_1)}{a_3 \cdot \mathcal{N}(\mathcal{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}), \boldsymbol{a})}. \end{aligned}$$

Step 7. Find the elliptical rotation angle using

$$\cos\theta = \frac{\mathcal{B}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})}{\sqrt{\mathcal{N}(\boldsymbol{x},\boldsymbol{a})}\sqrt{\mathcal{N}(\boldsymbol{y},\boldsymbol{a})}}$$

and define $C = \cos\theta$ and $S = \sin\theta$ where $S = \sqrt{1-C^2}$.

Step 8. Find the elliptical rotation matrix that rotates $\boldsymbol{x} = (x_1, y_1, y_1)$ to $\boldsymbol{y} = (x_2, y_2, z_2)$ elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ using Rodrigues matrix

$$R(\boldsymbol{u}, C, S, \Delta) = \begin{bmatrix} a_1u_1^2 + (1-a_1u_1^2)C & -\frac{\Delta u_3 S}{a_1} - a_2u_1u_2(C-1) & \frac{\Delta u_2 S}{a_1} - a_3u_1u_3(C-1) \\ \frac{\Delta u_3 S}{a_2} - a_1u_1u_2(C-1) & a_2u_2^2 + (1-a_2u_2^2)C & -\frac{\Delta u_1 S}{a_2} - a_3u_2u_3(C-1) \\ -\frac{\Delta u_2 S}{a_3} - a_1u_1u_3(C-1) & \frac{\Delta u_1 S}{a_3} - a_2u_2u_3(C-1) & a_3u_3^2 + (1-a_3u_3^2)C \end{bmatrix}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$.

Step 9. Define the matrix

$$\mathcal{H}(\mathbf{v},\boldsymbol{a}) = \frac{1}{\mathcal{N}(\mathbf{v},\boldsymbol{a})}\begin{bmatrix} \mathcal{N}(\mathbf{v},\boldsymbol{a}) - 2a_1v_1^2 & -2a_2v_1v_2 & -2a_3v_1v_3 \\ -2a_1v_2v_1 & \mathcal{N}(\mathbf{v},\boldsymbol{a}) - 2a_2v_2^2 & -2a_3v_2v_3 \\ -2a_1v_3v_1 & -2a_2v_3v_2 & \mathcal{N}(\mathbf{v},\boldsymbol{a}) - 2a_3v_3^2 \end{bmatrix}$$

for a given $\boldsymbol{v} = (v_1, v_2, v_3)$.

Step 10. Find the elliptical rotation matrix that rotates $\boldsymbol{x} = (x_1, y_1, y_1)$ to $\boldsymbol{y} = (x_2, y_2, z_2)$ elliptically on the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$ using Householder matrices

$$R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}) = \mathcal{H}(\boldsymbol{y},\boldsymbol{a})\mathcal{H}(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{a}).$$

Step 11. Define the set of Elliptic Quaternions $\mathbb{H}_{a_1,a_2,a_3} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ with

$$\mathbf{i}^2 = -a_1, \quad \mathbf{j}^2 = -a_2, \quad \mathbf{k}^2 = -a_3$$

and

$$\mathbf{ij} = \frac{\Delta}{a_3}\mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \frac{\Delta}{a_1}\mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \frac{\Delta}{a_2}\mathbf{j} = -\mathbf{ik}.$$

Step 12. Find $c = \cos\dfrac{\theta}{2} = \sqrt{\dfrac{\cos\theta + 1}{2}}$ and $s = \sqrt{1-c^2}$. Define the quaternion

$$q = \cos\frac{\theta}{2} + \boldsymbol{u}\sin\frac{\theta}{2} = c + su_1\mathbf{i} + su_2\mathbf{j} + su_3\mathbf{k}$$

where θ is the elliptical rotation angle and $\boldsymbol{u} = (u_1, u_2, u_3) = u_1i + u_2j + u_3k$ is the rotation axis obtained in Step 6 and Step 7.

Step 13. Find the elliptical rotation matrix that rotates $\boldsymbol{x}$ to $\boldsymbol{y}$ elliptically on the ellipsoid using the matrix

$$R(q,\boldsymbol{a},\Delta) = \begin{bmatrix} q_0^2+a_1q_1^2-a_2q_2^2-a_3q_3^2 & 2a_2q_1q_2-2\frac{q_0q_3\Delta}{a_1} & 2a_3q_1q_3+2\frac{q_0q_2\Delta}{a_1} \\ 2a_1q_1q_2+2\frac{q_0q_3\Delta}{a_2} & q_0^2-a_1q_1^2+a_2q_2^2-a_3q_3^2 & 2a_3q_2q_3-2\frac{q_0q_1\Delta}{a_2} \\ 2a_1q_1q_3-2\frac{q_0q_2\Delta}{a_3} & 2a_2q_2q_3+2\frac{q_0q_1\Delta}{a_3} & q_0^2-a_1q_1^2-a_2q_2^2+a_3q_3^2 \end{bmatrix}$$

corresponding to $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$.

Remark 7. In the $n > 3$ dimensional spaces, rotations can be classified such as simple, composite, and isoclinic, depending on plane of rotation. Simple rotation is a rotation with only one plane of rotation. In a simple rotation, there is a fixed plane. The rotation is said to take place about this plane. So points do not change their distance from this plane as they rotate. Orthogonal to this fixed plane is the plane of rotation. The rotation is said to take place in this plane. On the other hand, a rotation with two or more planes of rotation is called a composite rotation. The rotation can take place in each plane of rotation. These planes are orthogonal. In $\mathbb{R}^4$ it is called a double rotation. A double rotation has two angles of rotation, one for each plane of rotation. The rotation is specified by giving the two planes and two non-zero angles β and θ (if either angle is zero, then the rotation is simple). Finally, the isoclinic rotation is a special case of the composite rotation, when the two angles are equal [23]. In the 4 dimensional Euclidean and Lorentzian spaces, a skew symmetric matrix is decomposed as $A = \theta_1 A_1 + \theta_2 A_2$ using two skew-symmetric matrices A_1 and A_2 satisfying the properties $A_1A_2 = 0$, $A_1^3 = -A_1$ and $A_2^3 = -A_2$. Hence, the Rodrigues and Cayley rotation formulas can be used to generate 4 dimensional rotation matrices [6,7,9,10,13,20].

References

[1] Aragón-González, G., Aragón, J.L., Rodríguez-Andrade, M.A.: The decomposition of an orthogonal transformation as a product of reflections. J. Math. Phys. **47**, Art. No. 013509 (2006)

[2] Brackett, R.W.: Robotic manipulators and the product of exponentials formula. In: Mathematical Theory and Networks and Systems. Proceeding of International Symposium, pp. 120–127. Springer, Berlin (1984)

[3] Bükçü, B.: On the rotation matrices in semi-Euclidean space. Commun. Fac. Sci. Univ. Ank. Ser. A **1**(55), 7–13 (2006)

[4] Cayley, A.: Sur Quelques Proprietes des Determinants Gauches. The Collected Papers of Arthur Cayley SC.D.F.R.S. Cambridge University Press, Cambridge (1889)

[5] Cipra, B.A.: The Best of the 20th Century: Editors Name Top 10 Algorithms. SIAM News, vol. 33, no. 4 (2000)

[6] Eberly, D.: Constructing Rotation Matrices Using Power Series. Geometric Tools LLC (2007). http://geometrictools.com/

[7] Erdoğdu, M., Özdemir, M.: Cayley formula on matrix in Minkowski space-time. Int. J. Geom. Methods Mod. Phys. **12**(5) (2015). doi:10.1142/s0219887815500589

[8] Fuller, C.: A constructive proof of the Cartan–Dieudonné–Scherk Theorem in the real or complex case. J. Pure Appl. Algebra **215**, 1116–1126 (2011)

[9] Gallier, J.H.: Geometric methods and applications. For computer science and engineering. In: Texts in Applied Mathematics, vol. 38, 680p (2011)

[10] Gallier, J.: Remarks on the Cayley representation of orthogonal matrices and perturbing the diagonal of a matrix to make it invertible (2013). arXiv:math/0606320v2

[11] Gallier, J.H.: Notes on Differential Geometry and Lie Groups. University of Pennsylvania, Philadelphia, 730p (2014)

[12] Gallier, J., Xu, D.: Computing exponentials of skew symmetric matrices and logarithms of orthogonal matrices. Int. J. Robot. Autom. **18**, 10–20 (2000)

[13] Jadczyk, A., Szulga, J.: A comment on "On the rotation matrix in Minkowski space-time" by Özdemir and Erdoğdu. Rep. Math. Phys. **74**, 39–44 (2014)

[14] Kula, L., Karacan, M.K., Yaylı, Y.: Formulas for the exponential of semi symmetric matrix of order 4. Math. Comput. Appl. **10**, 99–104 (2005)

[15] Mackey, D.S., Mackey, N., Tisseur, F.: G-reflectors: analogues of householder transformations in scalar product spaces. Linear Algebra Appl. **385**, 187–213 (2004)

[16] Mebius, J.E.: Derivation of Euler–Rodrigues formula for three-dimensional rotations from the general formula for four dimensional rotations (2007). arXiv:math.GM

[17] Murray, M.R., Li, Z., Sastry, S.S.: A Mathematical Introduction to Robotic Manipulation. CRC Press, Boca Raton (1994)

[18] Norris, A.N.: Euler–Rodrigues and Cayley formulae for rotation of elasticity tensors. Math. Mech. Solids **13**, 465–498 (2008)

[19] Özdemir, M., Ergin, A.A.: Rotations with unit timelike quaternions in Minkowski 3-space. J. Geometry Phys. **56**, 322–336 (2006)

[20] Özdemir, M., Erdoğdu, M.: On the rotation matrix in Minkowski space-time. Rep. Math. Phys. **74**, 27–38 (2014)

[21] Özkaldı, S., Gündoğan, H.: Cayley formula, Euler parameters and rotations in 3-dimensional Lorentzian space. Adv. Appl. Clifford Algebras **20**, 367–377 (2010)

[22] Özdemir, M., Erdoğdu, M., Şimşek, H.: On the eigenvalues and eigenvectors of a Lorentzian rotation matrix by using split quaternions. Adv. Appl. Clifford Algebras **24**, 179–192 (2014)

[23] Pertti, L.: Clifford Algebras and Spinors. Cambridge University Press, Cambridge (2001). ISBN:978-0-521-00551-7

[24] Politi, T.: A formula for the exponential of a real skew-symmetric matrix of order 4. BIT Numer. Math. **41**, 842–845 (2001)

[25] Reventós Tarrida, A.: Affine Maps, Euclidean Motions and Quadrics. Springer, Berlin, 411 p (2011)

[26] Rodríguez-Andrade, M.A., Aragón-González, G., Aragón, J.L., Verde-Star, L.: An algorithm for the Cartan–Dieudonné theorem on generalized scalar product spaces. Linear Algebra Appl. **434**(5), 1238–1254 (2011)

[27] Schmidt, J., Nieman, H.: Using quaternions for parametrizing 3-D rotations in unconstrained nonlinear optimization. In: Vision Modeling and Visualization. Stuttgart, Germany, pp. 399–406 (2001)

[28] Selig, J.M.: Cayley maps for SE(3). In: 12th IFToMM World Congress, Besancon, pp. 18–21 (2007)

[29] Uhlig, F.: Constructive ways for generating (generalized) real orthogonal matrices as products of (generalized) symmetries. Linear Algebra Appl. **332–334**, 459–467 (2001)

[30] Vicci, L.: Quaternions and Rotations in 3-Space, The Algebra and Geometric Interpretation. Microelectric Systems Laboratory, Department of Computer Sciences, UNC (2001): TR01-014

[31] Weiner, J.L., Wilkens, G.R.: Quaternions and rotations in $\mathbb{E}^4$. Am. Math. Mon., pp. 69–76 (2005)

Mustafa Özdemir
Department of Mathematics
Akdeniz University
Antalya
Turkey
e-mail: mozdemir@akdeniz.edu.tr

Received: April 17, 2015.
Accepted: August 18, 2015.