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De- Moivre's and Euler Formulas for Matrices of Split Quaternions

Melek Erdoğdu[1] and Mustafa Özdemir[2]

[1] Department of Mathematics-Computer Sciences, Necmettin Erbakan University, 42060 Konya, Turkey

[2] Department of Mathematics, Akdeniz University, 07058 Antalya, Turkey
⋆ *Corresponding Author*. E-mail: merdogdu@konya.edu.tr

Abstract. In this paper, real matrix representations of split quaternions are examined in terms of the casual character of quaternion. Then, we give De-Moivre' s formula for real matrices of timelike and spacelike split quaternions, separately. Finally, we state the Euler Theorem for real matrices of pure split quaternions.

1 Introduction

Sir William Rowan Hamilton discovered the quaternions in 1843, which was one of the his best contribution made to mathematical science. This discovery is a way of extending complex numbers to higher dimensions. So the set of quaternions, which was introduced by Hamilton, can be represented as

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = -1 \text{ and } ijk = -1.$$

The set of quaternions is a member of noncommutative division algebra [9].

We know that real quaternion algebra is isomorphic to a real 4×4 matrix algebra. The real matrix representations of quaternions are investigated in [15], [6] and [7]. Euler and De-Moivre formulas for real matrices associated with quaternions are studied in [8].

In 1849, James Cockle introduced the set of split quaternions, also known as coquaternions. The real algebra of split quaternions, denoted by $\widehat{\mathbb{H}}$, is a four dimensional vector space over the real field $\mathbb{R}$ of real numbers with a basis $\{1, i, j, k\}$ satisfying

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$$i^2 = -1,\ j^2 = k^2 = 1 \text{ and } ijk = 1.$$

The set of split quaternions is noncommutative, too. Unlike quaternion algebra, the set of split quaternions contains zero divisor, nilpotent elements and nontrivial idempotents [10], [13].

Split quaternions is a recently developing topic. There are some studies related to geometric applications of split quaternions such as [10], [11], [14] and [12]. Particularly, the geometric and physical applications of quaternions require solving quaternionic equations. Therefore, there are many studies on quaternionic and split quaternionic equations. For example; the rearrangement method of solving two sided linear split quaternionic equations is given in [5] and De Moivre's formula is used to find the roots of split quaternion in work [13]. Furthermore, 2×2 complex matrix representation of any split quaternion is presented in [1] and the left and right real matrix representations of split quaternions are studied in [10].

In this paper, we will investigate real matrix representations of split quaternions. First, we present a brief introduction of split quaternions. Then, we examine real matrices associated with split quaternions depending on the casual character of the quaternion. Moreover, we give De-Moivre' s formula for real matrices of timelike and spacelike split quaternions, separately. Finally, we state the Euler theorem for real matrices of pure split quaternions.

2 Preliminaries

In this part, we present an introduction to split quaternions for the necessary background [10], [13], [14].

2.1 Split quaternions

The set split quaternions can be represented as

$$\widehat{\mathbb{H}} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\}$$

where the product table is:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

We write any split quaternion in the form $q = q_0 + q_1 i + q_2 j + q_3 k = S_q + \overrightarrow{V_q}$ where $S_q = q_0$ denotes the scalar part of q and $\overrightarrow{V_q} = q_1 i + q_2 j + q_3 k$ denotes vector part of q. If $S_q = 0$ then q is called pure split quaternion. The set of pure split quaternions are identified with the Minkowski 3 space. The Minkowski 3 space is Euclidean 3 space with the Lorentzian inner product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{\mathbb{L}} = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$ and denoted by $\mathbb{E}_1^3$. We say that a vector $\overrightarrow{u}$ in $\mathbb{E}_1^3$ is spacelike, timelike or null if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} > 0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} < 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{\mathbb{L}} = 0$, respectively. The

conjugate of a split quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is denoted by $\overline{q}$ and it is $\overline{q} = S_q - \overrightarrow{V_q} = q_0 - q_1 i - q_3 j - q_4 k$. For any $p, q \in \widehat{\mathbb{H}}$, the sum and product of split quaternions p and q are

$$p+q = S_p + S_q + \overrightarrow{V_p} + \overrightarrow{V_q},$$
$$pq = S_p S_q + \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle_{\mathbb{L}} + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \times_{\mathbb{L}} \overrightarrow{V_q},$$

respectively. Here $\times_{\mathbb{L}}$ denotes Lorentzian vector product and is defined as

$$\overrightarrow{u} \times_{\mathbb{L}} \overrightarrow{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

for vectors $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$ of Minkowski 3 space. And norm of the split quaternion q is defined by

$$N_q = \sqrt{|q\overline{q}|} = \sqrt{|q_0^2 + q_1^2 - q_2^2 - q_3^2|}.$$

It is essential to note that this is a semi norm since it is not positive definite. If $N_q = 1$ then q is called unit split quaternion and q/N_q is a unit split quaternion for $N_q \neq 0$. And the product

$$I_q = q\overline{q} = \overline{q}q = q_0^2 + q_1^2 - q_2^2 - q_3^2$$

determines the character of a split quaternion. A split quaternion is spacelike, timelike or lightlike (null) if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively. Polar forms of split quaternions are defined as follows:

i. Every timelike split quaternion with spacelike vector part can be written in the form

$$q = N_q(\cosh\theta + \overrightarrow{\varepsilon}\sinh\theta)$$

where $\overrightarrow{\varepsilon}$ is spacelike unit vector in $\mathbb{E}_1^3$.

ii. Every timelike split quaternion with timelike vector part can be written in the form

$$q = N_q(\cos\theta + \overrightarrow{\varepsilon}\sin\theta)$$

where $\overrightarrow{\varepsilon}$ is timelike unit vector in $\mathbb{E}_1^3$.

iii. Every spacelike split quaternion can be written in the form

$$q = N_q(\sinh\theta + \overrightarrow{\varepsilon}\cosh\theta)$$

where $\overrightarrow{\varepsilon}$ is spacelike unit vector in $\mathbb{E}_1^3$.

2.2 De-Moivre's formula for split quaternions

Euler and De-Moivre formula for complex numbers are generalized for quaternions in [2], [3]. Similarly to the discussions in [2] and [3], De-Moivre formula and roots of split quaternions are investigated in [13] by considering the casual characters of the quaternion. De-Moivre formulas for split quaternions, which are stated in [13], can be given as follows;

i. If $q = N_q(\cosh\theta + \overrightarrow{\varepsilon}\sinh\theta)$ be a timelike split quaternion with spacelike vector part then

$$q^n = (N_q)^n(\cosh n\theta + \overrightarrow{\varepsilon}\sinh n\theta),$$

ii. If $q = N_q(\cos\theta + \overrightarrow{\varepsilon}\sin\theta)$ be a timelike split quaternion with timelike vector part then

$$q^n = (N_q)^n(\cos n\theta + \overrightarrow{\varepsilon}\sin n\theta),$$

iii. If $q = N_q(\sinh\theta + \overrightarrow{\varepsilon}\cosh\theta)$ be a spacelike split quaternion then

$$q^n = \left\{ \begin{array}{l} (N_q)^n(\sinh n\theta + \overrightarrow{\varepsilon}\cosh n\theta) : n \text{ is odd} \\ \\ (N_q)^n(\cosh n\theta + \overrightarrow{\varepsilon}\sinh n\theta) : n \text{ is even} \end{array} \right\}$$

for $n \in \mathbb{N}$.

2.3 4×4 Real matrix representations of split quaternions

For any $q \in \widehat{\mathbb{H}}$, consider the linear map

$$g_q : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$$

defined as

$$g_q(p) = qp.$$

For $q = q_0 + q_1 i + q_2 j + q_3 k \in \widehat{\mathbb{H}}$, we have

$$g_q(1) = q_0 + q_1 i + q_2 j + q_3 k,$$

$$g_q(i) = -q_1 + q_0 i + q_3 j - q_2 k,$$

$$g_q(j) = q_2 + q_3 i + q_0 j + q_1 k,$$

$$g_q(k) = q_3 - q_2 i - q_1 j + q_0 k.$$

Using this map, we may define an isomorphism between $\widehat{\mathbb{H}}$ and algebra of the matrices

$$\left\{ \begin{bmatrix} q_0 & -q_1 & q_2 & q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} : q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}.$$

The above obtained 4×4 real matrix for the split quaternion q is denoted by

$$\mathbf{L}_q = \begin{bmatrix} q_0 & -q_1 & q_2 & q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$$

and called the left matrix representation of q.

Proposition 2.1. [10] For any $p,q \in \widehat{\mathbb{H}}$ and $r \in \mathbb{R}$, the followings are satisfied;

i. $\mathbf{L}_{p+q} = \mathbf{L}_p + \mathbf{L}_q$,
ii. $\mathbf{L}_{pq} = \mathbf{L}_p \mathbf{L}_q$,
iii. $\mathbf{L}_{rp} = r\mathbf{L}_p$,
iv. $\mathbf{L}_1 = \mathbf{I}_4$.

Similar to the previous discussion, consider the linear map

$$f_q : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$$

defined as

$$f_q(p) = pq.$$

For $q = q_0 + q_1 i + q_2 j + q_3 k \in \widehat{\mathbb{H}}$, we have

$$f_q(1) = q_0 + q_1 i + q_2 j + q_3 k,$$

$$f_q(i) = -q_1 + q_0 i - q_3 j + q_2 k,$$

$$f_q(j) = q_2 - q_3 i + q_0 j - q_1 k,$$

$$f_q(k) = q_3 + q_2 i + q_1 j + q_0 k.$$

By this map, we may define an isomorphism between $\widehat{\mathbb{H}}$ and algebra of the matrices

$$\left\{ \begin{bmatrix} q_0 & -q_1 & q_2 & q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} : q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}.$$

We denote above corresponding 4×4 real matrix for any split quaternion q by

$$\mathbf{R}_q = \begin{bmatrix} q_0 & -q_1 & q_2 & q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

and it is called the right matrix representation of q.

Proposition 2.2. [10] For any $p,q \in \widehat{\mathbb{H}}$ and $r \in \mathbb{R}$, the followings are satisfied;

i. $\mathbf{R}_{p+q} = \mathbf{R}_p + \mathbf{R}_q$,
ii. $\mathbf{R}_{pq} = \mathbf{R}_q \mathbf{R}_p$,
iii. $\mathbf{R}_{rp} = r\mathbf{R}_p$,
iv. $\mathbf{R}_1 = \mathbf{I}_4$.

3 De Moivre's formula for real matrices of split quaternions

Depending on the casual character of split quaternion, we can express real matrix representations of quaternion as follows:

i. Every timelike split quaternion $q. = N_q(\cosh\theta + \overrightarrow{\varepsilon}\sinh\theta)$ with spacelike vector part, we may write

$$\mathbf{L}_q = N_q \begin{bmatrix} \cosh\theta & -u_1\sinh\theta & u_2\sinh\theta & u_3\sinh\theta \\ u_1\sinh\theta & \cosh\theta & u_3\sinh\theta & -u_2\sinh\theta \\ u_2\sinh\theta & u_3\sinh\theta & \cosh\theta & -u_1\sinh\theta \\ u_3\sinh\theta & -u_2\sinh\theta & u_1\sinh\theta & \cosh\theta \end{bmatrix} = N_q[\cosh\theta\,\mathbf{I}_4 + \sinh\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

$$\mathbf{R}_q = N_q \begin{bmatrix} \cosh\theta & -u_1\sinh\theta & u_2\sinh\theta & u_3\sinh\theta \\ u_1\sinh\theta & \cosh\theta & -u_3\sinh\theta & u_2\sinh\theta \\ u_2\sinh\theta & -u_3\sinh\theta & \cosh\theta & u_1\sinh\theta \\ u_3\sinh\theta & u_2\sinh\theta & -u_1\sinh\theta & \cosh\theta \end{bmatrix} = N_q[\cosh\theta\,\mathbf{I}_4 + \sinh\theta\,\mathbf{R}_{\overrightarrow{\varepsilon}}]$$

where $\overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ is spacelike unit vector in $\mathbb{E}_1^3$.

ii. Every timelike split quaternion $q{=}N_q(\cos\theta + \overrightarrow{\varepsilon}\sin\theta)$ with timelike vector part, we may write

$$\mathbf{L}_q = N_q \begin{bmatrix} \cos\theta & -u_1\sin\theta & u_2\sin\theta & u_3\sin\theta \\ u_1\sin\theta & \cos\theta & u_3\sin\theta & -u_2\sin\theta \\ u_2\sin\theta & u_3\sin\theta & \cos\theta & -u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix} = N_q[\cos\theta\,\mathbf{I}_4 + \sin\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

$$\mathbf{R}_q = N_q \begin{bmatrix} \cos\theta & -u_1\sin\theta & u_2\sin\theta & u_3\sin\theta \\ u_1\sin\theta & \cos\theta & -u_3\sin\theta & u_2\sin\theta \\ u_2\sin\theta & -u_3\sin\theta & \cos\theta & u_1\sin\theta \\ u_3\sin\theta & u_2\sin\theta & -u_1\sin\theta & \cos\theta \end{bmatrix} = N_q[\cos\theta\,\mathbf{I}_4 + \sin\theta\,\mathbf{R}_{\overrightarrow{\varepsilon}}]$$

where $\overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ is timelike unit vector in $\mathbb{E}_1^3$.

iii. Every spacelike split quaternion $q = N_q(\sinh\theta + \overrightarrow{\varepsilon}\cosh\theta)$, we may write

$$\mathbf{L}_q = N_q \begin{bmatrix} \sinh\theta & -u_1\cosh\theta & u_2\cosh\theta & u_3\cosh\theta \\ u_1\cosh\theta & \sinh\theta & u_3\cosh\theta & -u_2\cosh\theta \\ u_2\cosh\theta & u_3\cosh\theta & \sinh\theta & -u_1\cosh\theta \\ u_3\cosh\theta & -u_2\cosh\theta & u_1\cosh\theta & \sinh\theta \end{bmatrix} = N_q[\sinh\theta\,\mathbf{I}_4 + \cosh\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

$$\mathbf{R}_q = N_q \begin{bmatrix} \sinh\theta & -u_1\cosh\theta & u_2\cosh\theta & u_3\cosh\theta \\ u_1\cosh\theta & \sinh\theta & -u_3\cosh\theta & u_2\cosh\theta \\ u_2\cosh\theta & -u_3\cosh\theta & \sinh\theta & u_1\cosh\theta \\ u_3\cosh\theta & u_2\cosh\theta & -u_1\cosh\theta & \sinh\theta \end{bmatrix} = N_q[\sinh\theta\,\mathbf{I}_4 + \cosh\theta\,\mathbf{R}_{\overrightarrow{\varepsilon}}]$$

where $\overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ is spacelike unit vector in $\mathbb{E}_1^3$.

Theorem 3.1. For any timelike split quaternion $q = N_q(\cosh\theta + \overrightarrow{\varepsilon}\sinh\theta)$ with spacelike vector part, the followings are satisfied;

i.

$$[\mathbf{L}_q]^n = (N_q)^n \begin{bmatrix} \cosh n\theta & -u_1 \sinh n\theta & u_2 \sinh n\theta & u_3 \sinh n\theta \\ u_1 \sinh n\theta & \cosh n\theta & u_3 \sinh n\theta & -u_2 \sinh n\theta \\ u_2 \sinh n\theta & u_3 \sinh n\theta & \cosh n\theta & -u_1 \sinh n\theta \\ u_3 \sinh n\theta & -u_2 \sinh n\theta & u_1 \sinh n\theta & \cosh n\theta \end{bmatrix} = (N_q)^n [\cosh n\theta \mathbf{I}_4 + \sinh n\theta \mathbf{L}_{\overrightarrow{\varepsilon}}],$$

ii.

$$[\mathbf{R}_q]^n = (N_q)^n \begin{bmatrix} \cosh n\theta & -u_1 \sinh n\theta & u_2 \sinh n\theta & u_3 \sinh n\theta \\ u_1 \sinh n\theta & \cosh n\theta & -u_3 \sinh n\theta & u_2 \sinh n\theta \\ u_2 \sinh n\theta & -u_3 \sinh n\theta & \cosh n\theta & u_1 \sinh n\theta \\ u_3 \sinh n\theta & u_2 \sinh n\theta & -u_1 \sinh n\theta & \cosh n\theta \end{bmatrix} = (N_q)^n [\cosh n\theta \, \mathbf{I}_4 + \sinh n\theta \, \mathbf{R}_{\overrightarrow{\varepsilon}}]$$

where $\overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ *is spacelike unit vector in* $\mathbb{E}_1^3$.

Proof. We use induction on positive integer n. Let $q{=}N_q(\cosh\theta + \overrightarrow{\varepsilon} \sinh\theta)$ be any timelike split quaternion where $\overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ is spacelike unit vector in $\mathbb{E}_1^3$.
i. Assume that

$$[\mathbf{L}_q]^n = (N_q)^n \begin{bmatrix} \cosh n\theta & -u_1 \sinh n\theta & u_2 \sinh n\theta & u_3 \sinh n\theta \\ u_1 \sinh n\theta & \cosh n\theta & u_3 \sinh n\theta & -u_2 \sinh n\theta \\ u_2 \sinh n\theta & u_3 \sinh n\theta & \cosh n\theta & -u_1 \sinh n\theta \\ u_3 \sinh n\theta & -u_2 \sinh n\theta & u_1 \sinh n\theta & \cosh n\theta \end{bmatrix}$$

$$= (N_q)^n [\cosh n\theta \, \mathbf{I}_4 + \sinh n\theta \, \mathbf{L}_{\overrightarrow{\varepsilon}}],$$

holds. Then

$(\mathbf{L}_q)^{n+1} = (\mathbf{L}_q)^n \, \mathbf{L}_q$

$$= N_q{}^{n+1} \begin{bmatrix} \cosh n\theta & -u_1 \sinh n\theta & u_2 \sinh n\theta & u_3 \sinh n\theta \\ u_1 \sinh n\theta & \cosh n\theta & u_3 \sinh n\theta & -u_2 \sinh n\theta \\ u_2 \sinh n\theta & u_3 \sinh n\theta & \cosh n\theta & -u_1 \sinh n\theta \\ u_3 \sinh n\theta & -u_2 \sinh n\theta & u_1 \sinh n\theta & \cosh n\theta \end{bmatrix} \begin{bmatrix} \cosh\theta & -u_1 \sinh\theta & u_2 \sinh\theta & u_3 \sinh\theta \\ u_1 \sinh\theta & \cosh\theta & u_3 \sinh\theta & -u_2 \sinh\theta \\ u_2 \sinh\theta & u_3 \sinh\theta & \cosh\theta & -u_1 \sinh\theta \\ u_3 \sinh\theta & -u_2 \sinh\theta & u_1 \sinh\theta & \cosh\theta \end{bmatrix}$$

Using the equality

$$-u_1^2 + u_2^2 + u_3^2 = 1,$$

and the identities

$$\cosh\theta \cosh n\theta + \sinh\theta \sinh n\theta = \cosh(n+1)\theta,$$

$$\cosh\theta \sinh n\theta + \sinh\theta \cosh n\theta = \sinh(n+1)\theta,$$

we get

$$(\mathbf{L}_q)^{n+1}=(N_q)^{n+1}\begin{bmatrix} \cosh(n+1)\theta & -u_1\sinh(n+1)\theta & u_2\sinh(n+1)\theta & u_3\sinh(n+1)\theta \\ u_1\sinh(n+1)\theta & \cosh(n+1)\theta & u_3\sinh(n+1)\theta & -u_2\sinh(n+1)\theta \\ u_2\sinh(n+1)\theta & u_3\sinh(n+1)\theta & \cosh(n+1)\theta & -u_1\sinh(n+1)\theta \\ u_3\sinh(n+1)\theta & -u_2\sinh(n+1)\theta & u_1\sinh(n+1)\theta & \cosh(n+1)\theta \end{bmatrix}$$

$$=(N_q)^{n+1}\left\{\cosh(n+1)\theta\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\sinh(n+1)\theta\begin{bmatrix}0&-u_1&u_2&u_3\\u_1&0&u_3&-u_2\\u_2&u_3&0&-u_1\\u_3&-u_2&u_1&0\end{bmatrix}\right\}$$

$$=(N_q)^{n+1}\left[\cosh(n+1)\theta\,\mathbf{I}_4+\sinh(n+1)\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}\right].$$

ii. Similarly to the proof of property i, we have

$$(\mathbf{R}_q)^{n+1}=(\mathbf{R}_q)^{n+1}=(\mathbf{R}_q)^n\,\mathbf{R}_q$$

$$=(N_q)^{n+1}\begin{bmatrix} \cosh(n+1)\theta & -u_1\sinh(n+1)\theta & u_2\sinh(n+1)\theta & u_3\sinh(n+1)\theta \\ u_1\sinh(n+1)\theta & \cosh(n+1)\theta & -u_3\sinh(n+1)\theta & u_2\sinh(n+1)\theta \\ u_2\sinh(n+1)\theta & -u_3\sinh(n+1)\theta & \cosh(n+1)\theta & u_1\sinh(n+1)\theta \\ u_3\sinh(n+1)\theta & u_2\sinh(n+1)\theta & -u_1\sinh(n+1)\theta & \cosh(n+1)\theta \end{bmatrix}$$

$$=(N_q)^{n+1}\left\{\cosh(n+1)\theta\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\sinh(n+1)\theta\begin{bmatrix}0&-u_1&u_2&u_3\\u_1&0&-u_3&u_2\\u_2&-u_3&0&u_1\\u_3&u_2&-u_1&0\end{bmatrix}\right\}$$

$$=(N_q)^{n+1}\left[\cosh(n+1)\theta\,\mathbf{I}_4+\sinh(n+1)\theta\,\mathbf{R}_{\overrightarrow{\varepsilon}}\right].$$

Theorem 3.2. For any timelike split quaternion $q=N_q(\cos\theta+\overrightarrow{\varepsilon}\sin\theta)$ with timelike vector part, then we have

i.

$$(\mathbf{L}_q)^n=(N_q)^n\begin{bmatrix} \cos n\theta & -u_1\sin n\theta & u_2\sin n\theta & u_3\sin n\theta \\ u_1\sin n\theta & \cos n\theta & u_3\sin n\theta & -u_2\sin n\theta \\ u_2\sin n\theta & u_3\sin n\theta & \cos n\theta & -u_1\sin n\theta \\ u_3\sin n\theta & -u_2\sin n\theta & u_1\sin n\theta & \cos n\theta \end{bmatrix}=(N_q)^n[\cos n\theta\,\mathbf{I}_4+\sin n\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

ii.

$$(\mathbf{R}_q)^n=(N_q)^n\begin{bmatrix} \cos n\theta & -u_1\sin n\theta & u_2\sin n\theta & u_3\sin n\theta \\ u_1\sin n\theta & \cos n\theta & -u_3\sin n\theta & u_2\sin n\theta \\ u_2\sin n\theta & -u_3\sin n\theta & \cos n\theta & u_1\sin n\theta \\ u_3\sin n\theta & u_2\sin n\theta & -u_1\sin n\theta & \cos n\theta \end{bmatrix}=(N_q)^n[\cos n\theta\,\mathbf{I}_4+\sin n\theta\,\mathbf{R}_{\overrightarrow{\varepsilon}}]$$

where $\overrightarrow{\varepsilon}=(u_1,u_2,u_3)$ *is timelike unit vector in* $\mathbb{E}_1^3$.

Proof. Using the equality

$$-u_1^2+u_2^2+u_3^2=-1,$$

and the identities

$$\cos\theta\cos n\theta-\sin\theta\sin n\theta=\cos(n+1)\theta,$$

$$\cos\theta\sin n\theta+\sin\theta\cos n\theta=\sin(n+1)\theta,$$

the proof of this theorem can be done by induction similar to the proof of Theorem 3.

Theorem 3.3. For any spacelike split quaternion $q=N_q(\sinh\theta+\overrightarrow{\varepsilon}\cosh\theta)$,

i. *If n odd then*

$$(\mathbf{L}_q)^n=(N_q)^n\begin{bmatrix}\sinh n\theta & -u_1\cosh n\theta & u_2\cosh n\theta & u_3\cosh n\theta\\ u_1\cosh n\theta & \sinh n\theta & u_3\cosh n\theta & -u_2\cosh n\theta\\ u_2\cosh n\theta & u_3\cosh n\theta & \sinh n\theta & -u_1\cosh n\theta\\ u_3\cosh n\theta & -u_2\cosh n\theta & u_1\cosh n\theta & \sinh n\theta\end{bmatrix}=(N_q)^n[\sinh n\theta\,\mathbf{I}_4+\cosh n\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

$$(\mathbf{R}_q)^n=(N_q)^n\begin{bmatrix}\sinh n\theta & -u_1\cosh n\theta & u_2\cosh n\theta & u_3\cosh n\theta\\ u_1\cosh n\theta & \sinh n\theta & -u_3\cosh n\theta & u_2\cosh n\theta\\ u_2\cosh n\theta & -u_3\cosh n\theta & \sinh n\theta & u_1\cosh n\theta\\ u_3\cosh n\theta & u_2\cosh n\theta & -u_1\cosh n\theta & \sinh n\theta\end{bmatrix}=(N_q)^n[\sinh n\theta\,\mathbf{I}_4+\cosh n\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}].$$

ii. *If n is even then*

$$(\mathbf{L}_q)^n=(N_q)^n\begin{bmatrix}\cosh n\theta & -u_1\sinh n\theta & u_2\sinh n\theta & u_3\sinh n\theta\\ u_1\sinh n\theta & \cosh n\theta & u_3\sinh n\theta & -u_2\sinh n\theta\\ u_2\sinh n\theta & u_3\sinh n\theta & \cosh n\theta & -u_1\sinh n\theta\\ u_3\sinh n\theta & -u_2\sinh n\theta & u_1\sinh n\theta & \cosh n\theta\end{bmatrix}=(N_q)^n[\cosh n\theta\,\mathbf{I}_4+\sinh n\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}],$$

$$(\mathbf{R}_q)^n=(N_q)^n\begin{bmatrix}\cosh n\theta & -u_1\sinh n\theta & u_2\sinh n\theta & u_3\sinh n\theta\\ u_1\sinh n\theta & \cosh n\theta & -u_3\sinh n\theta & u_2\sinh n\theta\\ u_2\sinh n\theta & -u_3\sinh n\theta & \cosh n\theta & u_1\sinh n\theta\\ u_3\sinh n\theta & u_2\sinh n\theta & -u_1\sinh n\theta & \cosh n\theta\end{bmatrix}=(N_q)^n[\cosh n\theta\,\mathbf{I}_4+\sinh n\theta\,\mathbf{L}_{\overrightarrow{\varepsilon}}].$$

where $\overrightarrow{\varepsilon}=(u_1,u_2,u_3)$ *is spacelike unit vector in* $\mathbb{E}_1^3$.

Proof. Similar to the proof of Theorem 3, using the equality

$$-u_1^2+u_2^2+u_3^2=1,$$

and the identities

$$\cosh\theta\cosh n\theta+\sinh\theta\sinh n\theta=\cosh(n+1)\theta,$$

$$\cosh\theta\sinh n\theta+\sinh\theta\cosh n\theta=\sinh(n+1)\theta,$$

this theorem can be easily proved by induction for each case, separately.

4 Euler formulas for real matrices of split quaternions

In this part, we will state Euler formulas for real matrices of spacelike and timelike pure split quaternions, separately.

Theorem 4.1. For any timelike pure unit split quaternion $q = \overrightarrow{\varepsilon} = (u_1, u_2, u_3)$, we have

$$e^{\theta \mathbf{L}_q} = \cos\theta \mathbf{I}_4 + \sin\theta\, \mathbf{L}_q \text{ and } e^{\theta \mathbf{R}_q} = \cos\theta \mathbf{I}_4 + \sin\theta\, \mathbf{R}_q.$$

Proof. Let $q = \overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ be any timelike pure unit split quaternion, we get

$$\mathbf{L}_q = \begin{bmatrix} 0 & -u_1 & u_2 & u_3 \\ u_1 & 0 & u_3 & -u_2 \\ u_2 & u_3 & 0 & -u_1 \\ u_3 & -u_2 & u_1 & 0 \end{bmatrix} \text{ and } (\mathbf{L}_q)^2 = (-u_1^2 + u_2^2 + u_3^2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the identity

$$-u_1^2 + u_2^2 + u_3^2 = -1,$$

we get $(\mathbf{L}_q)^2 = -\mathbf{I}_4$. So, we have

$$\begin{aligned} e^{\theta \mathbf{L}_q} &= \mathbf{I}_4 + \theta \mathbf{L}_q + \frac{\theta^2}{2!}(\mathbf{L}_q)^2 + \frac{\theta^3}{3!}(\mathbf{L}_q)^3 + \frac{\theta^4}{4!}(\mathbf{L}_q)^4 + \frac{\theta^5}{5!}(\mathbf{L}_q)^5 + \frac{\theta^6}{6!}(\mathbf{L}_q)^6 + \cdots \\ &= \mathbf{I}_4 + \theta \mathbf{L}_q + \frac{\theta^2}{2!}(-\mathbf{I}_4) + \frac{\theta^3}{3!}(-\mathbf{L}_q) + \frac{\theta^4}{4!}(\mathbf{I}_4) + \frac{\theta^5}{5!}(\mathbf{L}_q) + \frac{\theta^6}{6!}(-\mathbf{I}_4) + \cdots \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots)\mathbf{I}_4 + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)\mathbf{L}_q \\ &= \cos\theta\, \mathbf{I}_4 + \sin\theta\, \mathbf{L}_q. \end{aligned}$$

It is easy to see $(\mathbf{R}_q)^2 = -\mathbf{I}_4$ and $e^{\theta \mathbf{R}_q} = \cos\theta \mathbf{I}_4 + \sin\theta\, \mathbf{R}_q$.

Theorem 4.2. For any spacelike unit pure split quaternion $q = \overrightarrow{\varepsilon} = (u_1, u_2, u_3)$, we have

$$e^{\theta \mathbf{L}_q} = \cosh\theta\, \mathbf{I}_4 + \sinh\theta\, \mathbf{L}_q \text{ and } e^{\theta \mathbf{R}_q} = \cosh\theta\, \mathbf{I}_4 + \sinh\theta\, \mathbf{R}_q$$

Proof. Let $q = \overrightarrow{\varepsilon} = (u_1, u_2, u_3)$ be any spacelike pure unit split quaternion, we get

$$(\mathbf{L}_q)^2 = (-u_1^2 + u_2^2 + u_3^2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the identity

$$-u_1^2 + u_2^2 + u_3^2 = 1,$$

we get $(\mathbf{L}_q)^2 = \mathbf{I}_4$. So, we have

$$e^{\theta \mathbf{L}_q} = \mathbf{I}_4 + \theta \mathbf{L}_q + \frac{\theta^2}{2!}(\mathbf{L}_q)^2 + \frac{\theta^3}{3!}(\mathbf{L}_q)^3 + \frac{\theta^4}{4!}(\mathbf{L}_q)^4 + \frac{\theta^5}{5!}(\mathbf{L}_q)^5 + \frac{\theta^6}{6!}(\mathbf{L}_q)^6 + \cdots$$

$$= \mathbf{I}_4 + \theta \mathbf{L}_q + \frac{\theta^2}{2!}\mathbf{I}_4 + \frac{\theta^3}{3!}\mathbf{L}_q + \frac{\theta^4}{4!}\mathbf{I}_4 + \frac{\theta^5}{5!}\mathbf{L}_q + \frac{\theta^6}{6!}\mathbf{I}_4 + \cdots$$

$$= (1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots)\mathbf{I}_4 + (\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)\mathbf{L}_q$$

$$= \cosh\theta\, \mathbf{I}_4 + \sinh\theta\, \mathbf{L}_q.$$

Similarly, it can be easily shown that $(\mathbf{R}_q)^2 = \mathbf{I}_4$ and $e^{\theta \mathbf{R}_q} = \cosh\theta \mathbf{I}_4 + \sinh\theta\, \mathbf{R}_q$.

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