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Noether symmetries in interacting Yang–Mills quintessence cosmology

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Abstract

We investigate an interacting Yang–Mills quintessence cosmology in a spatially flat FLRW background. The model contains a non-minimal curvature coupling $f(\xi)$, a Yang–Mills kinetic coupling $I(\xi)$, an effective self-interaction potential $V_{\rm eff}(\xi)$, and a phenomenological interaction function $F(a,\xi)$ between the vector sector and pressureless matter. We apply the Noether symmetry approximation to the model to determine both the form of the unknown functions in the Lagrangian and to reduce the dynamics. The analytical solutions obtained through Noether symmetry show that the models lead to an accelerated expanding Universe. Finally, representative numerical examples illustrate an accelerated power-law solution in Branch 2 and a Branch 4 solution in which the deceleration parameter changes sign during evolution. We also discuss the homogeneous stability of the obtained solutions in detail. The branches belong to a selected power-law class; Branch 3 is retained only as a formal classical sector because its Yang–Mills kinetic coupling is negative under the condition $h > 0$.

1. Introduction

One of the main problems of modern cosmology is to explain the current accelerated phase of the Universe. Observations of type Ia supernovae, the cosmic microwave background radiation, the large-scale structure of the Universe, and baryonic acoustic oscillations indicate that the expansion of the Universe has recently entered an accelerated phase [1–3]. The simplest theoretical description of this behavior is to add a cosmological constant to Einstein's equations. This framework, usually expressed in the form of the ΛCDM model, is phenomenologically very successful and explains much of the observational data with a small number of parameters [4]. However, the cosmological constant faces theoretical questions such as the origin of the very small amount of vacuum energy, the fine-tuning problem, and the simultaneity problem [5–8]. For this reason, in recent decades, several dynamical models of dark energy have been proposed in which the component responsible for the cosmic acceleration is time-varying rather than a constant [9–11].

Among the dynamical models of dark energy, scenarios based on scalar fields have a special place. The quintessence, phantom, k-essence, tachyon, chameleon, and scalar-tensor models are examples of these frameworks [12–16]. In these models, the scalar field can create a negative effective pressure through potential energy, non-local kinetic structure, or non-minimal coupling to geometry, and thus lead to cosmic acceleration [17]. In addition, in some models, the possibility of direct interaction between the dark energy and dark matter components is also considered [18, 19]. Such interactions can change the cosmic history, affect the transition from the matter-dominant phase to the accelerated phase, and make the effective behavior of the equation of state different from that of models without interaction [20, 21]. However, a large part of these studies are based on scalar degrees of freedom, while vector fields and especially non-Abelian fields can also play an important dynamical role in cosmology. The use of vector fields in cosmology presents a fundamental challenge [22–24]. A single homogeneous

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vector field usually chooses a preferred spatial direction and is therefore incompatible with the isotropy of the FLRW background [25–28]. To solve this problem, configurations such as triad vectors, random vector fields, or arrangements with internal symmetry have been introduced [29]. Among them, the so-called cosmic triad configuration allows the three vector components to be placed in space in such a way that the resulting energy-momentum tensor remains homogeneous and isotropic in the background plane [30]. This idea is particularly attractive for $SU(2)$ or Yang–Mills fields, since the internal index of the moduli group can be aligned with the spatial indices in a symmetric configuration. In such a case, the background dynamics of a non-Abelian field can be reduced to a single effective variable, while its origin still lies in a non-Abelian vector section.

A common difficulty in generalized dark-energy models is the freedom in choosing the potential, non-minimal couplings, kinetic functions, and matter interactions [31, 32]. Although phenomenological choices are useful for constructing cosmological models, they may leave the functional form of the theory weakly constrained [33]. The Noether symmetry approach provides a way to restrict this freedom from the dynamical structure of the minisuperspace system [34]. Since Noether's theorem relates continuous symmetries to conserved quantities [35], the existence of a Noether symmetry in a point-like cosmological Lagrangian can generate a cyclic variable and a conserved charge [36], which simplify the field equations and may lead to exact solutions [37, 38]. In this sense, the method can also be used as a selection criterion for the unknown functions of the model [39, 40]. This approach has been applied in scalar-tensor theories, $f(R)$, $f(T)$, and $f(Q)$ gravity, Gauss–Bonnet and Horndeski models, tachyonic, phantom and quintessence cosmologies, and anisotropic backgrounds [32, 41–49]. These studies show its usefulness in reducing arbitrary choices and identifying conserved structures [50]. However, applications to vector dark-energy models with a non-Abelian Yang–Mills sector and direct matter interaction are less developed. In the cosmic-triad configuration, the vector field can preserve FLRW isotropy at the homogeneous level [26, 27], while its interaction with pressureless matter can modify the effective pressure of the total system. This motivates the present use of Noether symmetry for an interacting Yang–Mills model with non-minimal coupling and exact minisuperspace solutions [51, 52].

The interaction function $F(a,\xi)$ is introduced phenomenologically at the level of the homogeneous effective action, following the minisuperspace treatment of interacting dark-energy models [52]. Its scale-factor dependence is understood within the FLRW reduction. The Noether condition restricts the interaction to forms compatible with the selected symmetry, but does not derive a microscopic matter–vector coupling or establish a unique covariant completion. Constructing such a completion is left for future work.

Earlier cosmic-triad Noether studies considered Maxwell-type field strengths with fixed kinetic normalization [30], while related vector models examined non-minimal curvature or torsion couplings [27, 28]. Interacting quintessence studies instead applied the symmetry construction to a scalar dark-energy field coupled to matter [52]. The present work combines an $SU(2)$ cosmic triad, a variable kinetic coupling, a non-minimal curvature coupling, and a reconstructed matter interaction within a separable power-law class. Its specific results are the corresponding algebraic branches and their intersections, the constrained exact background solutions, and a homogeneous stability analysis that includes the Hamiltonian constraint and perturbations of the conserved Noether charge.

The main aim of this work is to study an interacting Yang–Mills quintessence cosmology in which the dark-energy sector is represented by a non-Abelian vector field in the cosmic-triad configuration [26–28]. In this setting, the homogeneous dynamics of the $SU(2)$ vector sector is described by the effective variable $\xi = 3\phi^2$, where $\phi(t)$ is the common homogeneous amplitude of the three spatial vectors in the cosmic-triad configuration. The model contains a non-minimal coupling $f(\xi)$ to the Ricci scalar, a Yang–Mills kinetic function $I(\xi)$, an effective potential $V_{\rm eff}(\xi)$, and an interaction function $F(a,\xi)$ between the vector sector and pressureless matter. After deriving the point-like Lagrangian in the FLRW background, we obtain the Hamiltonian constraint and the Euler–Lagrange equations and rewrite them in terms of an effective fluid description. This formulation is then used to discuss the acceleration condition, the crossing of the effective boundary $\omega_{\rm eff} = -1$, and de Sitter or quasi-de Sitter regimes. The Noether symmetry method is applied in the minisuperspace (a,ξ) to constrain the admissible forms of the coupling, potential, and interaction functions. For the relevant symmetry branches, adapted variables are introduced, cyclic coordinates and conserved charges are identified, and exact cosmological solutions are obtained. The homogeneous stability of these solutions is also examined, and representative numerical examples are used to illustrate their background evolution.

The structure of the paper is as follows. In section 2, the interacting Yang–Mills quintessence model is introduced, and the point-like Lagrangian, field equations, effective-fluid quantities, and acceleration-related conditions are derived. In section 3, the Noether symmetry approach is applied, the algebraic branches are obtained, and the corresponding cyclic variables, conserved charges, and exact solutions

are presented. In section 4, the homogeneous stability of the Noether solutions is investigated within the same minisuperspace framework. In section 5, numerical examples are given for selected branches to show the qualitative background behavior of the model. Finally, section 6 summarizes the main results.

2. Interacting Yang–Mills quintessence cosmology

In this section, we construct an interacting dark-sector cosmological model in which an isotropic non-Abelian vector sector replaces the scalar degree of freedom of the usual interacting quintessence scenario. The basic idea is to preserve the minisuperspace structure of interacting quintessence, while replacing the canonical scalar kinetic term by the kinetic contribution of an $SU(2)$ vector multiplet arranged in a cosmic-triad configuration. We consider the action

$$\mathcal{A}=\int \mathrm{d}^4x\sqrt{-g}\left[f(\xi)\,R-\frac{1}{4}I(\xi)\sum_{a=1}^{3}F^{a}_{\mu\nu}F^{a\mu\nu}-V(\xi)+\widetilde{\mathcal{L}}_m\left(g_{\mu\nu},\xi\right)\right], \tag{1}$$

where R is the Ricci scalar, $I(\xi)$ is a general kinetic coupling function, and $V(\xi)$ is a self-interaction potential. The scalar quantity

$$\xi\equiv\sum_{a=1}^{3}A^{a}_{\mu}A^{a\mu} \tag{2}$$

is a Lorentz scalar constructed from the vector multiplet. The matter sector is assumed to be non-trivially coupled to the vector dark-energy sector through

$$\widetilde{\mathcal{L}}_m\left(g_{\mu\nu},\xi\right)=\mathcal{L}_m\left(g_{\mu\nu}\right)+\mathcal{L}_m^{\mathrm{int}}\left(g_{\mu\nu},\xi\right). \tag{3}$$

Here, $\mathcal{L}_m(g_{\mu\nu})$ represents the Lagrangian density of pressureless matter, for which we take $\mathcal{L}_m(g_{\mu\nu})=-Ma^{-3}$. The interaction part of the matter Lagrangian is assumed to have the form $\mathcal{L}_m^{\mathrm{int}}(g_{\mu\nu},\xi)=-Ma^{-3}F(a,\xi)$, where M is a positive constant associated with the matter density and has the dimension of energy density in natural units, while $F(a,\xi)$ is an arbitrary dimensionless function characterizing the dark-sector interaction. The expressions involving a specify the homogeneous matter sector; a microscopic covariant matter-interaction Lagrangian is not constructed here. The field-strength tensor of the $SU(2)$ vector multiplet is defined as

$$F^{a}_{\mu\nu}=\partial_\mu A^{a}_{\nu}-\partial_\nu A^{a}_{\mu}+g_{\mathrm{YM}}\epsilon^{abc}A^{b}_{\mu}A^{c}_{\nu}, \tag{4}$$

where g_{YM} is the non-Abelian coupling constant and ϵ^{abc} denotes the structure constants of $SU(2)$. We use the metric signature $(-,+,+,+)$ and units such that $c=\hbar=M_{\mathrm{Pl}}=1$. We assume a spatially flat Friedmann-Lemaitre-Robertson-Walker geometry,

$$\mathrm{d}s^2=-\mathrm{d}t^2+a^2\left(t\right)\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right), \tag{5}$$

where $a(t)$ is the scale factor and $H\equiv\dot{a}/a$ is the Hubble parameter. A single homogeneous vector field would generally select a preferred spatial direction and would therefore spoil isotropy. For this reason, we adopt the cosmic-triad configuration

$$A^{a}_{0}=0,\qquad A^{a}_{i}=\delta^{a}_{i}\,a\left(t\right)\phi\left(t\right), \tag{6}$$

which preserves homogeneity and isotropy at the background level. For this ansatz, one finds

$$\xi=3\phi^2\left(t\right). \tag{7}$$

Thus, the homogeneous vector sector can be described by the single background variable $\xi(t)$, with the restriction $\xi\geqslant 0$. In particular, the parametrization in terms of ξ is singular at $\xi=0$, whereas the original variable ϕ is regular there. Hence, the minisuperspace description below is to be understood on the branch $\xi>0$. Substituting the FLRW metric and the cosmic-triad ansatz into the action (1), and removing total-derivative terms from the Einstein–Hilbert part, we obtain the point-like Lagrangian

$$\mathcal{L}=-6\left(f(\xi)-\frac{1}{12}I(\xi)\,\xi\right)a\dot{a}^2-6\left(f'\left(\xi\right)-\frac{1}{12}I(\xi)\right)a^2\dot{a}\dot{\xi}+\frac{a^3I(\xi)}{8\xi}\dot{\xi}^2-a^3V_{\mathrm{eff}}-M[1+F(a,\xi)]. \tag{8}$$

where

$$V_{\rm eff} \equiv V(\xi) + \frac{1}{6} I(\xi) g_{YM}^2 \xi^2, \tag{9}$$

and a prime denotes differentiation with respect to ξ. The restriction to $\xi > 0$ stated after (7) is reflected in the coefficient $I(\xi)/\xi$ in (8). The Jacobian of the map (7) vanishes at $\phi = 0$, so this map is not locally invertible there. This is a limitation of the ξ coordinate; regularity of a particular model at the origin also depends on its coupling and potential functions. By introducing $h(\xi) \equiv f(\xi) - \frac{1}{12} I(\xi)\xi$, the point-like Lagrangian can be rewritten as

$$\mathcal{L} = -6h(\xi) a\dot{a}^2 - 6\left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right) a^2 \dot{a}\dot{\xi} + \frac{a^3 I(\xi)}{8\xi}\dot{\xi}^2 - a^3 V_{\rm eff}(\xi) - M[1 + F(a,\xi)]. \tag{10}$$

The Hamiltonian energy function associated with the point-like Lagrangian is

$$E_{\mathcal{L}} = \dot{a}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\xi}\frac{\partial \mathcal{L}}{\partial \dot{\xi}} - \mathcal{L}. \tag{11}$$

Using (10), and imposing the Hamiltonian constraint $E_{\mathcal{L}} = 0$, one obtains

$$3h(\xi) H^2 = -3\left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right) H\dot{\xi} + \frac{I(\xi)}{16\xi}\dot{\xi}^2 + \frac{1}{2} V_{\rm eff}(\xi) + \frac{M}{2a^3}[1 + F(a,\xi)]. \tag{12}$$

Variation of the point-like Lagrangian with respect to the scale factor $a(t)$ yields the following acceleration equation:

$$\begin{aligned} h(\xi)\left(2\dot{H} + 3H^2\right) = &-2h'(\xi) H\dot{\xi} - \left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right)\ddot{\xi} \\ &- \left(h''(\xi) + \frac{1}{12}\left(I'(\xi) + \xi I''(\xi)\right) + \frac{I(\xi)}{16\xi}\right)\dot{\xi}^2 + \frac{1}{2} V_{\rm eff}(\xi) + \frac{M}{6a^2}\frac{\partial F}{\partial a}. \end{aligned} \tag{13}$$

The equation of motion for the variable $\xi(t)$ is obtained by varying the point-like Lagrangian with respect to ξ. This gives

$$\begin{aligned} &\ddot{\xi} + 3H\dot{\xi} + \frac{1}{2}\left(\frac{I'(\xi)}{I(\xi)} - \frac{1}{\xi}\right)\dot{\xi}^2 - \frac{24\xi}{I(\xi)}\left[\left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right)\dot{H} + \left(2h'(\xi) + \frac{1}{4}\xi I'(\xi)\right) H^2\right] \\ &+ \frac{4\xi}{I(\xi)}\left[V'_{\rm eff}(\xi) + \frac{M}{a^3}\frac{\partial F}{\partial \xi}\right] = 0. \end{aligned} \tag{14}$$

2.1. Effective fluid description

The Hamiltonian constraint can then be written in the standard Friedmann form $3H^2 = \rho_{\rm eff}$, provided $h(\xi) \neq 0$, where the total effective energy density is

$$\rho_{\rm eff} = \frac{1}{h(\xi)}\left[-3\left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right) H\dot{\xi} + \frac{I(\xi)}{16\xi}\dot{\xi}^2 + \frac{1}{2} V_{\rm eff}(\xi) + \frac{M}{2a^3}(1 + F(a,\xi))\right]. \tag{15}$$

Similarly, using the corrected scale-factor equation, the second Friedmann equation is written as $2\dot{H} + 3H^2 = -p_{\rm eff}$, with the effective pressure

$$\begin{aligned} p_{\rm eff} = \frac{1}{h(\xi)}\Bigg[&2h'(\xi) H\dot{\xi} + \left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right)\ddot{\xi} \\ &+ \left(h''(\xi) + \frac{1}{12}\left[I'(\xi) + \xi I''(\xi)\right] + \frac{I(\xi)}{16\xi}\right)\dot{\xi}^2 - \frac{1}{2} V_{\rm eff}(\xi) - \frac{M}{6a^2}\frac{\partial F(a,\xi)}{\partial a}\Bigg]. \end{aligned} \tag{16}$$

The effective equation-of-state parameter is defined by

$$\omega_{\rm eff} \equiv \frac{p_{\rm eff}}{\rho_{\rm eff}}. \tag{17}$$

Using equations (15) and (16), one obtains

$$\omega_{\rm eff} = \frac{2h'(\xi) H\dot{\xi} + \left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right)\ddot{\xi} + \left[h''(\xi) + \frac{1}{12}\left(I'(\xi) + \xi I''(\xi)\right) + \frac{I(\xi)}{16\xi}\right]\dot{\xi}^2 - \frac{1}{2} V_{\rm eff}(\xi) - \frac{M}{6a^2}\frac{\partial F(a,\xi)}{\partial a}}{-3\left(h'(\xi) + \frac{1}{12}\xi I'(\xi)\right) H\dot{\xi} + \frac{I(\xi)}{16\xi}\dot{\xi}^2 + \frac{1}{2} V_{\rm eff}(\xi) + \frac{M}{2a^3}[1 + F(a,\xi)]}. \tag{18}$$

This expression is well defined provided that $\rho_{\rm eff} \neq 0$.

2.2. Late-time acceleration and cosmological evolution

The effective description introduced above allows one to discuss the accelerated phase directly in terms of the total equation-of-state parameter. Since the Hamiltonian constraint has been written in the standard form $3H^2 = \rho_{\rm eff}$, the acceleration of the scale factor is controlled by

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\rho_{\rm eff}\left(1+3\omega_{\rm eff}\right). \tag{19}$$

Therefore, on an expanding branch with $\rho_{\rm eff} > 0$, accelerated expansion occurs whenever

$$\omega_{\rm eff} < -\frac{1}{3}. \tag{20}$$

A quasi-de Sitter phase corresponds to $\omega_{\rm eff} \simeq -1$, or equivalently $|\dot{H}| \ll H^2$. Thus, the model can account for the recent accelerated expansion if the combination of the Yang–Mills effective potential, the non-minimal coupling terms and the interaction contribution in (18) drives the effective pressure to sufficiently negative values at late times.

For a more explicit form of the acceleration criterion, it is useful to introduce

$$\mathcal{B}(\xi) \equiv h'(\xi) + \frac{1}{12}\xi I'(\xi). \tag{21}$$

Using equations (15) and (16), and assuming the physical branch $h(\xi) > 0$, the condition $\rho_{\rm eff} + 3p_{\rm eff} < 0$ becomes

$$3\mathcal{B}\ddot{\xi} + 3\left(2h' - \mathcal{B}\right)H\dot{\xi} + \left(3\mathcal{B}' + \frac{I}{4\xi}\right)\dot{\xi}^2 - V_{\rm eff} + \frac{M}{2a^3}\left(1+F\right) - \frac{M}{2a^2}\frac{\partial F}{\partial a} < 0. \tag{22}$$

This condition shows that the effective potential contributes in the direction of acceleration, while the ordinary dust contribution tends to decelerate the expansion. The interaction term can either help or oppose acceleration depending on the sign of $\frac{\partial F}{\partial a}$. In particular, for $\frac{\partial F}{\partial a} > 0$, the interaction produces a negative pressure contribution and can reduce the value of $\omega_{\rm eff}$. The kinetic and non-minimal derivative terms determine whether the transition is of quintessence type, phantom-like, or close to a de Sitter regime.

A simple late-time regime is obtained when the field evolves slowly on cosmological time scales. In this case the terms proportional to $\dot{\xi}$, $\ddot{\xi}$, and $\dot{\xi}^2$ in (22) are subdominant. The acceleration condition then reduces to

$$V_{\rm eff} + \frac{M}{2a^2}\frac{\partial F}{\partial a} > \frac{M}{2a^3}\left(1+F\right). \tag{23}$$

For vanishing interaction this gives the usual requirement that the potential energy of the dark sector must dominate over the matter density by a sufficient amount. In the interacting case, however, the second term on the left-hand side of (23) can assist the onset of acceleration. Hence, even if $V_{\rm eff}$ is not by itself large enough to dominate at the present epoch, a suitable interaction function with $\frac{\partial F}{\partial a} > 0$ can generate a negative effective pressure and trigger the transition from deceleration to acceleration.

The condition for crossing the cosmological-constant boundary is also transparent in this formulation. From $\rho_{\rm eff} + p_{\rm eff} = 0$, one obtains

$$\mathcal{B}\ddot{\xi} - \left(h' + \frac{1}{4}\xi I'\right)H\dot{\xi} + \left(\mathcal{B}' + \frac{I}{8\xi}\right)\dot{\xi}^2 + \frac{M}{2a^3}\left(1+F\right) - \frac{M}{6a^2}\frac{\partial F}{\partial a} = 0. \tag{24}$$

This relation indicates that the potential cancels out of the exact $\omega_{\rm eff} = -1$ condition. Therefore, the approach to or crossing of the phantom divide is controlled by the kinetic sector, the non-minimal coupling functions, and the interaction pressure. For $I > 0$, a purely kinetic contribution tends to oppose phantom-like behavior, while the mixed terms proportional to $H\dot{\xi}$ and the interaction term proportional to $\frac{\partial F}{\partial a}$ may allow $\omega_{\rm eff}$ to become smaller than -1 without requiring a negative Yang–Mills kinetic coupling.

A particularly important possibility is a late-time fixed point with constant $\xi = \xi_*$ and nearly constant $H = H_*$. If the matter and interaction contributions dilute asymptotically, equations (12) and (14) give

$$H_*^2 = \frac{V_{\rm eff}(\xi_*)}{6h(\xi_*)}, \tag{25}$$

together with the algebraic condition

$$h(\xi_*)\,V'_{\rm eff}(\xi_*) - \left[2h'(\xi_*) + \frac{1}{4}\xi_* I'(\xi_*)\right] V_{\rm eff}(\xi_*) = 0. \tag{26}$$

Thus, a de Sitter solution exists if there is a positive root $\xi_* > 0$ satisfying (26), with $h(\xi_*) > 0$ and $V_{\rm eff}(\xi_*) > 0$. In this case the Yang–Mills self-interaction, including the non-Abelian contribution contained in $V_{\rm eff}$, acts as an effective dark-energy source. If the fixed point is stable under homogeneous perturbations, the solution can describe the asymptotic future of the Universe. The interaction sector can also support a de Sitter-like phase. If the interaction contribution does not dilute but instead approaches a constant effective density, then asymptotically it must satisfy

$$a\frac{\partial F}{\partial a} \simeq 3\,(1+F)\,. \tag{27}$$

In that case the interaction behaves as a cosmological-constant component at the background level. More generally, if at fixed ξ the interaction grows as $F(a,\xi) \propto a^s$, then the interaction-dominated equation of state is approximately

$$\omega_{\rm int} \simeq -\frac{s}{3}. \tag{28}$$

Consequently, $s > 1$ gives accelerated expansion, $s = 3$ gives a cosmological-constant-like contribution, and $s > 3$ gives a phantom-like effective behavior. For such an interaction, the transition from a matter-dominated era to an accelerated era occurs approximately when

$$(s-1)\,F(a_{\rm tr},\xi_{\rm tr}) \simeq 1, \tag{29}$$

provided that the field-velocity terms and the potential contribution are subdominant near the transition.

These conditions show how a consistent background history can be obtained. At early times, if $F \ll 1$, $h(\xi)$ is slowly varying, and the terms involving $\dot{\xi}$ are small compared with the dust term in (15), the model admits an effectively matter-dominated stage with $\omega_{\rm eff} \simeq 0$. As the Universe expands, the dust density decreases as a^{-3}, while the potential part of the Yang–Mills sector or a growing interaction contribution can become dominant. The transition occurs when (22) changes sign. At late times, the system approaches accelerated expansion if either the field settles near a stable solution satisfying (26), or the interaction sector satisfies a scaling close to (27). Therefore, the present accelerated expansion can be reproduced in this framework, but it is not automatic; it requires a physical branch with $h > 0$, a positive total effective energy density, and either a sufficiently dominant positive $V_{\rm eff}$ or an interaction function whose scale-factor dependence produces negative effective pressure at late times.

3. Noether symmetry approach

Starting from the point-like Lagrangian (10), we now apply the Noether symmetry method in the minisuperspace $\mathcal{Q} = \{a,\xi\}$. The first prolongation of a point transformation on the tangent space $T\mathcal{Q} = \{a,\xi,\dot{a},\dot{\xi}\}$ is written as

$$X = \alpha\,(a,\xi)\,\frac{\partial}{\partial a} + \beta\,(a,\xi)\,\frac{\partial}{\partial \xi} + \dot{\alpha}\,(a,\xi)\,\frac{\partial}{\partial \dot{a}} + \dot{\beta}\,(a,\xi)\,\frac{\partial}{\partial \dot{\xi}}, \tag{30}$$

where

$$\dot{\alpha} = \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial \xi}\dot{\xi}, \qquad \dot{\beta} = \frac{\partial \beta}{\partial a}\dot{a} + \frac{\partial \beta}{\partial \xi}\dot{\xi}. \tag{31}$$

We denote the configuration-space generator by $\widetilde{X} = \alpha\partial_a + \beta\partial_\xi$, so that the vector field in (30) is its first prolongation, $X = \widetilde{X}^{[1]}$. Adapted coordinates satisfy $\widetilde{X}z = 1$ and $\widetilde{X}v = 0$. In these coordinates $\widetilde{X} = \partial_z$, and (32) makes z cyclic in the transformed Lagrangian; v is constant along the symmetry orbits, but need not be constant in time.

In the absence of a gauge term, the Noether symmetry condition is

$$X\mathcal{L} = 0. \tag{32}$$

Since the Lagrangian (10) is quadratic in the velocities, equation (32) gives a polynomial in $\dot{a}$ and $\dot{\xi}$. Therefore, the coefficients of $\dot{a}^2$, $\dot{a}\dot{\xi}$, $\dot{\xi}^2$, and the velocity-independent part must vanish separately. Using the definition $\mathcal{B}(\xi) \equiv h'(\xi) + \frac{1}{12}\xi I'(\xi)$, introduced in equation (21), the determining system takes the form

$$h\alpha + ah'\beta + 2ah\frac{\partial\alpha}{\partial a} + a^2\mathcal{B}\frac{\partial\beta}{\partial a} = 0, \tag{33}$$

$$2\mathcal{B}\alpha + a\mathcal{B}'\beta + 2h\frac{\partial\alpha}{\partial\xi} + a\mathcal{B}\left(\frac{\partial\alpha}{\partial a} + \frac{\partial\beta}{\partial\xi}\right) - \frac{a^2 I}{24\xi}\frac{\partial\beta}{\partial a} = 0, \tag{34}$$

$$\frac{3a^2 I}{8\xi}\alpha + \frac{a^3}{8}\left(\frac{I'}{\xi} - \frac{I}{\xi^2}\right)\beta - 6a^2\mathcal{B}\frac{\partial\alpha}{\partial\xi} + \frac{a^3 I}{4\xi}\frac{\partial\beta}{\partial\xi} = 0, \tag{35}$$

and

$$\alpha\left(3a^2 V_{\rm eff} + M\frac{\partial F}{\partial a}\right) + \beta\left(a^3 V'_{\rm eff} + M\frac{\partial F}{\partial\xi}\right) = 0. \tag{36}$$

Equations (33)–(36) constrain the unknown functions $\alpha(a,\xi)$, $\beta(a,\xi)$, $h(\xi)$, $I(\xi)$, $V_{\rm eff}(\xi)$, and $F(a,\xi)$. Hence, the existence of a Noether symmetry selects admissible functional forms for the non-minimal coupling, the Yang–Mills kinetic coupling, the effective potential, and the interaction term. The conserved Noether charge associated with the generator (30) is obtained from

$$\Sigma = \alpha\frac{\partial\mathcal{L}}{\partial\dot{a}} + \beta\frac{\partial\mathcal{L}}{\partial\dot{\xi}}. \tag{37}$$

For the Lagrangian (10), this gives

$$\Sigma = -\left(12ha\alpha + 6a^2\mathcal{B}\beta\right)\dot{a} + \left(\frac{a^3 I}{4\xi}\beta - 6a^2\mathcal{B}\alpha\right)\dot{\xi}. \tag{38}$$

The conservation law $\mathrm{d}\Sigma/\mathrm{d}t = 0$ provides a first integral of the minisuperspace dynamics. Therefore, once a solution of the determining system is specified, the cosmological equations can be reduced by using the corresponding cyclic variable associated with the Noether symmetry. We now impose the power-law ansatz for the effective potential,

$$V_{\rm eff}(\xi) = V_0\xi^n, \qquad V_0 \neq 0, \qquad n \neq 0. \tag{39}$$

Following the separable Noether construction, we take

$$\alpha(a,\xi) = \alpha_0 a^\sigma \xi^\ell, \qquad \beta(a,\xi) = \mu\alpha_0 a^{\sigma-1}\xi^{\ell+1}, \qquad \alpha_0 \neq 0. \tag{40}$$

To close the determining system, we assume compatible power-law forms for the functions appearing in the point-like Lagrangian,

$$h(\xi) = h_0\xi^m, \qquad I(\xi) = \iota_0\xi^{m-1}. \tag{41}$$

With this choice,

$$\mathcal{B}(\xi) = \left(mh_0 + \frac{(m-1)\,\iota_0}{12}\right)\xi^{m-1}. \tag{42}$$

The choices in (39)–(41) define a particular solvable class of the determining system, rather than its general solution. On the domain $a > 0$ and $\xi > 0$, the relative powers in the generator components and coupling functions allow a common monomial factor to be removed from each kinetic determining equation. The remaining conditions therefore reduce to algebraic relations among the exponents and amplitudes. The effective-potential ansatz is a compatible choice within this construction; it is not uniquely imposed by Noether symmetry while the interaction remains free. Other functional forms or generators may admit additional symmetries and are not excluded by this analysis.

Substitution of (40)–(42) into (33)–(35), and removing the common non-vanishing factors, gives

$$h_0(1 + \mu m + 2\sigma) + \mu(\sigma - 1)\left[mh_0 + \frac{(m-1)\,\iota_0}{12}\right] = 0. \tag{43}$$

$$[2 + \mu(m-1) + \sigma + \mu(\ell+1)]\left[mh_0 + \frac{(m-1)\,\iota_0}{12}\right] + 2\ell h_0 - \frac{\mu(\sigma-1)\,\iota_0}{24} = 0. \tag{44}$$

and

$$-6\ell\left[mh_0+\frac{(m-1)\,\iota_0}{12}\right]+\iota_0\left[\frac{3}{8}+\frac{\mu\,(m-2)}{8}+\frac{\mu\,(\ell+1)}{4}\right]=0. \tag{45}$$

The velocity-independent equation (36) does not impose a direct relation between μ and n, because the interaction function $F(a,\xi)$ is still free. Instead, it reconstructs the admissible interaction. For the ansatz (40), the velocity-independent equation (36) reduces to

$$\frac{\partial F}{\partial a}+\frac{\mu\xi}{a}\frac{\partial F}{\partial \xi}=-\frac{V_0}{M}\,(3+\mu n)\,a^2\xi^n. \tag{46}$$

The characteristic invariant of this first-order equation is

$$w=\xi\,a^{-\mu}. \tag{47}$$

Therefore, the general solution can be written as

$$F(a,\xi)=\mathcal{F}\left(\xi\,a^{-\mu}\right)-\frac{V_0}{M}a^3\xi^n, \tag{48}$$

where $\mathcal{F}$ is an arbitrary function of its argument. In the special case $3+\mu n=0$, the source term in (46) vanishes. Then $a^3\xi^n$ is itself a function of the invariant $\xi\,a^{-\mu}$, and the second term in (48) can be absorbed into $\mathcal{F}$. Hence the interaction reduces to the homogeneous form

$$F(a,\xi)=\mathcal{F}\left(\xi\,a^{-\mu}\right). \tag{49}$$

Thus, the total potential term in the reduced Lagrangian depends only on the Noether invariant w,

$$a^3V_{\rm eff}\,(\xi)+M[1+F(a,\xi)]=M[1+\mathcal{F}_N(w)]\,. \tag{50}$$

where $\mathcal{F}_N$ denotes the Noether-invariant combination obtained after absorbing the potential contribution into the homogeneous interaction function. For obtaining explicit exact solutions, the remaining Noether-invariant part of the interaction is chosen in the power-law form

$$1+\mathcal{F}_N(w)=\Lambda w^{\nu},\qquad \Lambda\neq 0. \tag{51}$$

Equation (51) is a further choice made to obtain explicit solutions, since the symmetry condition leaves an arbitrary function of the invariant in (50).

It is useful to solve the algebraic system (43)–(45) in terms of the ratio between the two coupling amplitudes. We therefore introduce

$$r\equiv\frac{\iota_0}{h_0},\qquad \mathcal{C}\equiv m+\frac{(m-1)\,r}{12}, \tag{52}$$

where $h_0\neq 0$. In terms of these quantities, the common combination appearing in (43)–(45) is $h_0\mathcal{C}$. In the regular sector

$$2+\mu\mathcal{C}\neq 0,\qquad 24\mathcal{C}-\mu r\neq 0, \tag{53}$$

the first and third equations can be solved for σ and ℓ. This gives

$$\sigma=\frac{\mu\,(\mathcal{C}-m)-1}{2+\mu\mathcal{C}},\qquad \ell=\frac{r(3+\mu m)}{2\,(24\mathcal{C}-\mu r)}. \tag{54}$$

Substituting (54) into (44), and clearing the non-vanishing denominators specified in (53), one obtains the factorized condition

$$(r+12)\,(m\mu+3)\left[(m-1)^2\,r+12m^2\right]\left[\mu r\,(4m-\mu-4)+48\,(m\mu+1)\right]=0. \tag{55}$$

Thus, the Noether symmetry conditions split into four algebraic branches. These branches are defined as follows:

$$\text{Branch 1:}\qquad r=-12, \tag{56}$$

$$\text{Branch 2:} \qquad \mu = \frac{-3}{m}, \tag{57}$$

$$\text{Branch 3:} \qquad r = \frac{-12m^2}{(m-1)^2}, \tag{58}$$

$$\text{Branch 4:} \qquad r = -\frac{48\,(m\mu+1)}{\mu\,(4m-\mu-4)}. \tag{59}$$

It should be noted that the explicit forms of the algebraic branches are valid only when the corresponding denominators are non-vanishing. In particular, Branch 2 requires $m \neq 0$, Branch 3 requires $m \neq 1$, and Branch 4 requires $\mu(4m-\mu-4) \neq 0$; if these conditions are not satisfied, the corresponding factor in equation (55) must be analyzed directly. In Branch 4, this leads to the degenerate subcase $m = 1/2$ and $\mu = -2$, for which the fourth factor vanishes independently of r, subject to the regular-sector condition $r \neq -12$. In what follows, we analyze each of the admissible branches obtained from equation (55) in detail, taking into account their regularity conditions and possible degenerate subcases.

It is necessary to separate the algebraic branches obtained from the Noether determining equations from the sectors that remain physically admissible in the original four-dimensional theory. The first branch, given by equation (56), corresponds to $r = -12$. Using the definition of $h(\xi)$, this condition implies

$$f(\xi) = h(\xi) + \frac{1}{12} I(\xi)\,\xi = 0. \tag{60}$$

Therefore, the curvature coupling $f(\xi)R$ in the action (1) vanishes identically in this branch. As a result, this sector does not represent a non-minimally coupled Yang–Mills quintessence model with an active curvature contribution. Moreover, on the branch $\xi > 0$, the condition $h(\xi) > 0$ requires $h_0 > 0$, while $r = -12$ gives $I(\xi) < 0$. Hence Branch 1 cannot simultaneously satisfy a positive effective gravitational coupling and a positive Yang–Mills kinetic coupling. For these reasons, Branch 1 will be regarded only as an algebraic solution of the Noether system and will not be included in the construction of physical cosmological solutions or in the homogeneous stability analysis. In the following subsections, we restrict attention to the retained algebraic sectors of Branches 2–4. Branch 3 is included for its formal classical solutions, subject to the kinetic-sign limitation discussed below.

3.1. Branch 2

3.1.1. Generic constrained solution

We now consider Branch 2, defined by equation (57). Since this branch follows from the factor $m\mu + 3 = 0$, one has

$$\mu = -\frac{3}{m}, \qquad m \neq 0. \tag{61}$$

The coupling functions remain

$$h(\xi) = h_0 \xi^m, \qquad I(\xi) = r h_0 \xi^{m-1}, \qquad \mathcal{B}(\xi) = h_0 \mathcal{C} \xi^{m-1}, \qquad \mathcal{C} = m + \frac{(m-1)\,r}{12}. \tag{62}$$

For later use we define

$$\mathcal{K}_2 \equiv 32m^2 + \left(4m^2 - 4m + 3\right) r, \qquad \mathcal{D}_2 \equiv 24m^2 + \left(2m^2 - 2m + 3\right) r. \tag{63}$$

In this branch, the regular-sector conditions (53) become

$$4m + (m-1)\,r \neq 0, \qquad \mathcal{D}_2 \neq 0. \tag{64}$$

In the generic sector, after excluding the $r = -12$ sector and avoiding the intersections with Branches 3 and 4, we also assume

$$r \neq -12, \qquad (m-1)^2\, r + 12m^2 \neq 0, \qquad \mathcal{K}_2 \neq 0. \tag{65}$$

Using equation (54), Branch 2 gives

$$\sigma = 1, \qquad \ell = 0. \tag{66}$$

The corresponding adapted variables may be chosen as

$$v = \xi a^{-\mu} = \xi a^{3/m}, \qquad z = \frac{1}{\alpha_0}\ln a, \tag{67}$$

which satisfy $\widetilde{X}z = 1$ and $\widetilde{X}v = 0$. The inverse transformation is

$$a = \mathrm{e}^{\alpha_0 z}, \qquad \xi = v\mathrm{e}^{\mu\alpha_0 z} = v\mathrm{e}^{-3\alpha_0 z/m}. \tag{68}$$

Using equations (50) and (51), the point-like Lagrangian can be written in the reduced form

$$\mathcal{L}_{\mathrm{B2}} = h_0 v^m \left[A_2\dot{z}^2 + B_2\dot{z}\frac{\dot{v}}{v} + C_2\left(\frac{\dot{v}}{v}\right)^2\right] - M\Lambda v^\nu, \tag{69}$$

where the subscript B2 denotes the generic sector of Branch 2, characterized by $m\mu + 3 = 0$ together with the non-overlap conditions (65). The constants appearing in equation (69) are

$$A_2 = \frac{3\alpha_0^2}{8m^2}\mathcal{K}_2, \qquad B_2 = -\frac{\alpha_0}{4m}\mathcal{D}_2, \qquad C_2 = \frac{r}{8}. \tag{70}$$

Introducing

$$Y \equiv \frac{\dot{v}}{v}, \tag{71}$$

the Euler–Lagrange equation for the cyclic coordinate gives the conserved momentum

$$\Sigma_{\mathrm{B2}} = \frac{\partial\mathcal{L}_{\mathrm{B2}}}{\partial\dot{z}} = h_0 v^m\left(2A_2\dot{z} + B_2 Y\right). \tag{72}$$

The equation obtained by varying v in the reduced Lagrangian (69), using the Euler–Lagrange equation, is

$$B_2\ddot{z} + 2C_2\dot{Y} - mA_2\dot{z}^2 + mC_2Y^2 + \frac{M\Lambda\nu}{h_0}v^{\nu-m} = 0. \tag{73}$$

The Hamiltonian energy function associated with equation (69) is

$$E_{\mathrm{B2}} = h_0 v^m\left[A_2\dot{z}^2 + B_2\dot{z}Y + C_2Y^2\right] + M\Lambda v^\nu. \tag{74}$$

The Hamiltonian constraint is imposed by setting

$$E_{\mathrm{B2}} = 0. \tag{75}$$

For $A_2 \neq 0$, equation (72) gives

$$\dot{z} = \frac{\Sigma_{\mathrm{B2}}}{2A_2h_0v^m} - \frac{B_2}{2A_2}Y. \tag{76}$$

Substitution into equation (75) yields the first-order equation

$$Y^2 = -\frac{1}{h_0G_2}\left[\frac{\Sigma_{\mathrm{B2}}^2}{4A_2h_0}v^{-2m} + M\Lambda v^{\nu-m}\right], \tag{77}$$

where

$$G_2 \equiv C_2 - \frac{B_2^2}{4A_2} = -\frac{m^2(r+12)\left[(m-1)^2r + 12m^2\right]}{6\mathcal{K}_2}. \tag{78}$$

Thus, the generic constrained dynamics is determined by the quadrature

$$t - t_0 = \pm\int^v \frac{\mathrm{d}u}{u\left\{-\frac{1}{h_0G_2}\left[\frac{\Sigma_{\mathrm{B2}}^2}{4A_2h_0}u^{-2m} + M\Lambda u^{\nu-m}\right]\right\}^{1/2}}, \tag{79}$$

together with

$$z(t) = z_0 + \int^t\left[\frac{\Sigma_{\mathrm{B2}}}{2A_2h_0v(\tau)^m} - \frac{B_2}{2A_2}\frac{\dot{v}(\tau)}{v(\tau)}\right]\mathrm{d}\tau, \tag{80}$$

where z_0 is an integration constant. The scale factor and the field variable are then obtained from equation (68).

3.1.2. Zero-charge sector

A closed solution is obtained in the zero-charge sector $\Sigma_{\mathrm{B2}}=0$. In this case,

$$\dot{z}=-\frac{B_2}{2A_2}\frac{\dot{v}}{v},\qquad \left(\frac{\dot{v}}{v}\right)^2=-\frac{M\Lambda}{h_0G_2}v^{\nu-m}. \tag{81}$$

Defining

$$\kappa_2^2\equiv-\frac{M\Lambda}{h_0G_2}, \tag{82}$$

with $\kappa_2^2\geqslant 0$, one finds, for $\nu\neq m$,

$$v(t)=\left[v_0^{\frac{m-\nu}{2}}+\varepsilon\frac{m-\nu}{2}\kappa_2(t-t_0)\right]^{\frac{2}{m-\nu}},\qquad \varepsilon=\pm1, \tag{83}$$

where v_0 is an integration constant. For $\nu=m$, the corresponding solution is

$$v(t)=v_0\mathrm{e}^{\varepsilon\kappa_2(t-t_0)}. \tag{84}$$

The cyclic variable is

$$z(t)=z_0-\frac{B_2}{2A_2}\ln\left(\frac{v(t)}{v_0}\right). \tag{85}$$

Hence,

$$a(t)=a_0\left(\frac{v(t)}{v_0}\right)^{p_2},\qquad \xi(t)=\xi_0\left(\frac{v(t)}{v_0}\right)^{1-\frac{3p_2}{m}}, \tag{86}$$

where

$$p_2\equiv-\frac{\alpha_0B_2}{2A_2}=\frac{m\mathcal{D}_2}{3\mathcal{K}_2},\qquad \xi_0a_0^{3/m}=v_0. \tag{87}$$

3.1.3. Constant-invariant sector

The constant-invariant sector $v=v_0$ gives another closed solution. In this case, equations (73) and (75) imply

$$\nu=-m,\qquad z(t)=z_0+z_1(t-t_0),\qquad z_1^2=-\frac{M\Lambda}{h_0A_2}v_0^{-2m}. \tag{88}$$

The conserved momentum is then constrained by

$$\Sigma_{\mathrm{B2}}^2=-4h_0A_2M\Lambda=-\frac{3\alpha_0^2h_0M\Lambda}{2m^2}\mathcal{K}_2. \tag{89}$$

Using equation (68), the corresponding cosmological solution is

$$a(t)=a_0\mathrm{e}^{\lambda(t-t_0)},\qquad \xi(t)=\xi_0\mathrm{e}^{-3\lambda(t-t_0)/m},\qquad \xi_0a_0^{3/m}=v_0, \tag{90}$$

where

$$\lambda^2=-\frac{8m^2M\Lambda}{3h_0\mathcal{K}_2}v_0^{-2m}. \tag{91}$$

Thus, in the generic sector of Branch 2, the dynamics is reducible to the first-order form (77); the zero-charge sector gives the explicit solution (86), while the constant-invariant sector gives the exponential solution (90).

3.2. Branch 3

3.2.1. Couplings and physical limitation

We now consider Branch 3, defined by (58). In this branch one has

$$\iota_0 = -\frac{12m^2}{(m-1)^2}h_0, \qquad \mathcal{C} = -\frac{m}{m-1}, \tag{92}$$

and therefore

$$h(\xi) = h_0\xi^m, \qquad I(\xi) = -\frac{12m^2}{(m-1)^2}h_0\xi^{m-1}, \qquad \mathcal{B}(\xi) = -\frac{m}{m-1}h_0\xi^{m-1}. \tag{93}$$

For this branch, the original curvature coupling is reconstructed as

$$f(\xi) = h(\xi) + \frac{1}{12}I(\xi)\xi = h_0\xi^m\frac{1-2m}{(m-1)^2}. \tag{94}$$

Therefore, unlike the excluded $r=-12$ sector, Branch 3 has a non-vanishing curvature coupling for $m \neq 1/2$. However, if the condition $h(\xi) > 0$ is imposed on the branch $\xi > 0$, then $h_0 > 0$, and equation (93) gives $I(\xi) < 0$. Hence Branch 3 should be understood as an active-curvature sector with a wrong-sign Yang–Mills kinetic coupling. In the local high-frequency limit, the transverse Yang–Mills fluctuations retain the kinetic sign of $I(\xi)$ in (1). Consequently, the negative coupling gives negative-energy directions in their quadratic Hamiltonian and obstructs a perturbatively healthy quantum interpretation [53]. The degeneracy of the homogeneous reduced Lagrangian does not remove these inhomogeneous modes, and conditional homogeneous stability cannot establish their absence. Branch 3 is therefore retained only as a formal classical solution of the Noether system, not as a viable ghost-free dark-energy sector. An effective-theory interpretation alone does not resolve this problem; any proposed resolution would require a consistent modification of the theory and a full perturbation analysis. The condition $h > 0$ is a background restriction and does not by itself establish positivity of the physical perturbation kinetic terms.

It is useful to introduce

$$\Delta_3 \equiv m\mu - 2(m-1). \tag{95}$$

The regular-sector conditions (53) reduce to

$$m \neq 0, 1, \qquad \Delta_3 \neq 0. \tag{96}$$

The value $m = 1/2$ gives $r=-12$ and therefore belongs to the excluded algebraic sector discussed above. In the generic non-overlapping sector we also take

$$m \neq \frac{1}{2}, \qquad m\mu + 3 \neq 0. \tag{97}$$

3.2.2. Generic constrained solution

Using (54), one obtains

$$\sigma = \frac{m^2\mu + m - 1}{\Delta_3}, \qquad \ell = -\frac{m(m\mu+3)}{2\Delta_3}. \tag{98}$$

Since

$$1 - \sigma - \mu\ell = \frac{m\mu+3}{2}, \tag{99}$$

a convenient adapted coordinate system is

$$v = \xi a^{-\mu}, \qquad z = \frac{2}{\alpha_0(m\mu+3)}a^{1-\sigma}\xi^{-\ell}. \tag{100}$$

These variables satisfy

$$\widetilde{X}z = 1, \qquad \widetilde{X}v = 0. \tag{101}$$

Thus, z is the cyclic coordinate and v is the corresponding Noether invariant. The inverse transformation is

$$a=\left[\frac{\alpha_0\,(m\mu+3)}{2}z\right]^{\frac{2}{m\mu+3}}v^{-\frac{m}{\Delta_3}},\qquad \xi=\left[\frac{\alpha_0\,(m\mu+3)}{2}z\right]^{\frac{2\mu}{m\mu+3}}v^{-\frac{2(m-1)}{\Delta_3}}. \tag{102}$$

Using (50) and (51), the point-like Lagrangian reduces to

$$\mathcal{L}_{\rm B3}=-A_3v^{s_3}\dot{z}^2-M\Lambda v^{\nu}, \tag{103}$$

where

$$A_3\equiv\frac{3}{2}\alpha_0^2h_0\frac{\Delta_3^2}{(m-1)^2},\qquad s_3\equiv-\frac{m\,(2m+1)}{\Delta_3}. \tag{104}$$

The absence of $\dot{v}$ in (103) follows from the degeneracy of the kinetic sector in this branch.

The Euler–Lagrange equation for the cyclic coordinate gives the conserved momentum

$$\Sigma_{\rm B3}=\frac{\partial\mathcal{L}_{\rm B3}}{\partial\dot{z}}=-2A_3v^{s_3}\dot{z}. \tag{105}$$

The equation obtained by varying v in the reduced Lagrangian (103), using the Euler–Lagrange equation, is

$$A_3s_3v^{s_3-1}\dot{z}^2+M\Lambda\nu v^{\nu-1}=0. \tag{106}$$

The Hamiltonian energy function associated with (103) is

$$E_{\rm B3}=-A_3v^{s_3}\dot{z}^2+M\Lambda v^{\nu}. \tag{107}$$

The Hamiltonian constraint is imposed by setting

$$E_{\rm B3}=0. \tag{108}$$

Combining (106) with (108) gives

$$\nu=-s_3=\frac{m\,(2m+1)}{\Delta_3}. \tag{109}$$

The same constraint fixes the conserved momentum as

$$\Sigma_{\rm B3}^2=4A_3M\Lambda=6\alpha_0^2h_0\frac{\Delta_3^2}{(m-1)^2}M\Lambda. \tag{110}$$

Real solutions require the right-hand side of (110) to be non-negative. Since the invariant v is not fixed by the reduced equations, the constrained solution can be written as

$$z(t)=z_0-\frac{\Sigma_{\rm B3}}{2A_3}\int^t v(\tau)^{-s_3}\,{\rm d}\tau, \tag{111}$$

together with (102).

3.2.3. Constant-invariant scaling sector

A closed scaling solution is obtained by taking the invariant to be constant, $v=v_0$. Then

$$z(t)=z_0+z_1\,(t-t_0)\,,\qquad z_1^2=\frac{M\Lambda}{A_3}v_0^{-2s_3}=\frac{2M\Lambda\,(m-1)^2}{3\alpha_0^2h_0\Delta_3^2}v_0^{-2s_3}. \tag{112}$$

Substitution of (112) into (102) gives

$$a(t)=v_0^{-\frac{m}{\Delta_3}}\left[\frac{\alpha_0\,(m\mu+3)}{2}\left(z_0+z_1\,(t-t_0)\right)\right]^{\frac{2}{m\mu+3}}, \tag{113}$$

and

$$\xi(t) = v_0^{-\frac{2(m-1)}{\Delta_3}} \left[\frac{\alpha_0(m\mu+3)}{2}(z_0 + z_1(t-t_0))\right]^{\frac{2\mu}{m\mu+3}}. \tag{114}$$

After a shift of the origin of time, this solution can be written as

$$a(t) = a_3(t-t_*)^{\frac{2}{m\mu+3}}, \qquad \xi(t) = \xi_3(t-t_*)^{\frac{2\mu}{m\mu+3}}, \qquad \xi_3 a_3^{-\mu} = v_0. \tag{115}$$

3.2.4. Intersection with Branch 2

The limiting case $m\mu+3=0$ is the intersection of Branch 3 with Branch 2 and is not covered by (100). For $m \neq -1/2$, a regular adapted choice is

$$z = \frac{1}{\alpha_0}\ln a, \qquad v = \xi a^{3/m}, \qquad \mu = -\frac{3}{m}. \tag{116}$$

The inverse map is the same as (68). In this overlapping sector, the reduced Lagrangian becomes

$$\mathcal{L}_{\mathrm{B2}\cap\mathrm{B3}} = -\frac{3h_0}{2(m-1)^2} v^m \left[\alpha_0(2m+1)\dot{z} - m\frac{\dot{v}}{v}\right]^2 - M\Lambda v^\nu. \tag{117}$$

Defining

$$\mathcal{Q}_3 \equiv \alpha_0(2m+1)\dot{z} - m\frac{\dot{v}}{v}, \tag{118}$$

the conserved momentum is

$$\Sigma_{\mathrm{B2}\cap\mathrm{B3}} = -\frac{3h_0\alpha_0(2m+1)}{(m-1)^2} v^m \mathcal{Q}_3. \tag{119}$$

The equation obtained by varying v in the reduced Lagrangian (117), using the Euler–Lagrange equation, is

$$2\dot{\mathcal{Q}}_3 + 2m\frac{\dot{v}}{v}\mathcal{Q}_3 + \mathcal{Q}_3^2 + \frac{2(m-1)^2 M\Lambda\nu}{3h_0 m} v^{\nu-m} = 0. \tag{120}$$

The Hamiltonian energy function is

$$E_{\mathrm{B2}\cap\mathrm{B3}} = -\frac{3h_0}{2(m-1)^2} v^m \mathcal{Q}_3^2 + M\Lambda v^\nu. \tag{121}$$

Imposing $E_{\mathrm{B2}\cap\mathrm{B3}} = 0$ and taking the constant-invariant sector $v = v_0$, the constrained equations imply

$$\nu = -m, \qquad z(t) = z_0 + z_1(t-t_0), \qquad z_1^2 = \frac{2M\Lambda(m-1)^2}{3h_0\alpha_0^2(2m+1)^2} v_0^{-2m}. \tag{122}$$

The conserved momentum is then constrained by

$$\Sigma^2_{\mathrm{B2}\cap\mathrm{B3}} = \frac{6\alpha_0^2 h_0 M\Lambda(2m+1)^2}{(m-1)^2}. \tag{123}$$

Using (116), the corresponding cosmological solution is

$$a(t) = a_0 e^{\lambda(t-t_0)}, \qquad \xi(t) = \xi_0 e^{-3\lambda(t-t_0)/m}, \qquad \xi_0 a_0^{3/m} = v_0, \tag{124}$$

where

$$\lambda^2 = \frac{2M\Lambda(m-1)^2}{3h_0(2m+1)^2} v_0^{-2m}. \tag{125}$$

Thus, the generic regular sector of Branch 3 gives the power-law solution (115), while its intersection with Branch 2 gives the exponential solution (124).

3.3. Branch 4

3.3.1. Generic constrained solution

We now consider Branch 4, defined by (59). It is convenient to introduce

$$\Gamma_4 \equiv \mu + 4 - 4m, \qquad \Delta_4 \equiv m\mu - 2m + 2. \tag{126}$$

Then (59) can be written as

$$r = \frac{48\,(m\mu+1)}{\mu\Gamma_4}, \qquad \mathcal{C} = \frac{m\mu^2 + 4m - 4}{\mu\Gamma_4}. \tag{127}$$

The corresponding coupling functions are

$$h(\xi) = h_0\xi^m, \qquad I(\xi) = \frac{48\,(m\mu+1)}{\mu\Gamma_4}h_0\xi^{m-1}, \qquad \mathcal{B}(\xi) = h_0\mathcal{C}\xi^{m-1}. \tag{128}$$

The regular-sector conditions (53) imply

$$\mu \neq 0, \qquad \Gamma_4 \neq 0, \qquad \mu + 2 \neq 0, \qquad \Delta_4 \neq 0. \tag{129}$$

In the generic sector of Branch 4, which does not overlap with Branch 2, we also take

$$m\mu + 3 \neq 0. \tag{130}$$

Using (54), one obtains

$$\sigma = \frac{4m^2\mu - 4m\mu + 8m - \mu - 8}{(\mu+2)\,\Delta_4}, \qquad \ell = -\frac{(m\mu+1)\,(m\mu+3)}{(\mu+2)\,\Delta_4}. \tag{131}$$

These parameters satisfy

$$1 - \sigma - \mu\ell = m\mu + 3. \tag{132}$$

Therefore, a convenient adapted coordinate system is

$$v = \xi\, a^{-\mu}, \qquad z = \frac{1}{\alpha_0\,(m\mu+3)}a^{1-\sigma}\xi^{-\ell}. \tag{133}$$

The variables (133) satisfy

$$\widetilde{X}z = 1, \qquad \widetilde{X}v = 0. \tag{134}$$

Thus, z is the cyclic coordinate and v is the Noether invariant. The inverse transformation is

$$a = [\alpha_0\,(m\mu+3)\,z]^{\frac{1}{m\mu+3}}\,v^{-\frac{m\mu+1}{(\mu+2)\Delta_4}}, \qquad \xi = [\alpha_0\,(m\mu+3)\,z]^{\frac{\mu}{m\mu+3}}\,v^{\frac{\Gamma_4}{(\mu+2)\Delta_4}}. \tag{135}$$

Using (50) and (51), the point-like Lagrangian reduces to

$$\mathcal{L}_{\mathrm{B4}} = A_4 v^{s_4}\dot{z}\,Y - M\Lambda v^{\nu}, \qquad Y \equiv \frac{\dot{v}}{v}, \tag{136}$$

where

$$A_4 \equiv \frac{6\alpha_0 h_0\,(\mu+2)\,\Delta_4}{\mu\Gamma_4}, \qquad s_4 \equiv -\frac{4m^2 + 2m\mu - 4m + 3}{(\mu+2)\,\Delta_4}. \tag{137}$$

In this branch, the kinetic sector contains only the mixed term $\dot{z}\dot{v}/v$ in the adapted variables.

The Euler–Lagrange equation for the cyclic coordinate gives the conserved momentum

$$\Sigma_{\mathrm{B4}} = \frac{\partial\mathcal{L}_{\mathrm{B4}}}{\partial\dot{z}} = A_4 v^{s_4} Y. \tag{138}$$

The equation obtained by varying v in the reduced Lagrangian (136), using the Euler–Lagrange equation, is

$$A_4\ddot{z} + M\Lambda\nu v^{\nu-s_4} = 0. \tag{139}$$

The Hamiltonian energy function associated with (136) is

$$E_{\mathrm{B4}} = A_4 v^{s_4} \dot{z} Y + M\Lambda v^{\nu}. \tag{140}$$

The Hamiltonian constraint is imposed by setting

$$E_{\mathrm{B4}} = 0. \tag{141}$$

For $M\Lambda \neq 0$ and $v > 0$, the constrained system requires $\Sigma_{\mathrm{B4}} \neq 0$. Combining (138) with (141), one obtains

$$Y = \frac{\Sigma_{\mathrm{B4}}}{A_4 v^{s_4}}, \qquad \dot{z} = -\frac{M\Lambda}{\Sigma_{\mathrm{B4}}} v^{\nu}. \tag{142}$$

The first equation in (142) gives, for $s_4 \neq 0$,

$$v(t) = \left[v_0^{s_4} + \frac{s_4 \Sigma_{\mathrm{B4}}}{A_4} (t - t_0) \right]^{\frac{1}{s_4}}. \tag{143}$$

For $s_4 = 0$, the corresponding solution is

$$v(t) = v_0 \exp\left[\frac{\Sigma_{\mathrm{B4}}}{A_4} (t - t_0) \right]. \tag{144}$$

The cyclic variable is obtained from the second equation in (142). For $\nu + s_4 \neq 0$, it is

$$z(t) = z_0 - \frac{A_4 M\Lambda}{\Sigma_{\mathrm{B4}}^2 (\nu + s_4)} \left[v(t)^{\nu + s_4} - v_0^{\nu + s_4} \right]. \tag{145}$$

For $\nu + s_4 = 0$, one instead obtains

$$z(t) = z_0 - \frac{A_4 M\Lambda}{\Sigma_{\mathrm{B4}}^2} \ln\left(\frac{v(t)}{v_0} \right). \tag{146}$$

The scale factor and the field variable follow from (135). Defining

$$\mathcal{Z}_4(t) \equiv \alpha_0 (m\mu + 3) z(t), \tag{147}$$

one obtains

$$a(t) = \mathcal{Z}_4(t)^{\frac{1}{m\mu+3}} v(t)^{-\frac{m\mu+1}{(\mu+2)\Delta_4}}, \qquad \xi(t) = \mathcal{Z}_4(t)^{\frac{\mu}{m\mu+3}} v(t)^{\frac{\Gamma_4}{(\mu+2)\Delta_4}}. \tag{148}$$

Equations (143)–(148) give the constrained cosmological solution in the generic regular sector of Branch 4.In contrast with the degenerate Branch 3 sector, a constant-invariant solution with $v = v_0$ is not compatible with (141) when $M\Lambda \neq 0$.

3.3.2. Intersection with Branch 2

The limiting case $m\mu + 3 = 0$ lies at the intersection of Branch 4 with Branch 2 and is not covered by (133). In this case

$$\mu = -\frac{3}{m}, \qquad r = -\frac{32m^2}{\Pi_4}, \qquad \Pi_4 \equiv 4m^2 - 4m + 3, \tag{149}$$

with

$$m \neq 0, \qquad m \neq -\frac{1}{2}, \qquad m \neq \frac{3}{2}. \tag{150}$$

A regular adapted coordinate system is

$$v = \xi a^{3/m}, \qquad z = \frac{1}{\alpha_0} \ln a. \tag{151}$$

The inverse transformation is

$$a = \mathrm{e}^{\alpha_0 z}, \qquad \xi = v\mathrm{e}^{-3\alpha_0 z/m}. \tag{152}$$

The reduced Lagrangian becomes

$$\mathcal{L}_{\mathrm{B2}\cap\mathrm{B4}} = h_0 v^m \left[B_{24}\dot{z}\,Y + C_{24}\,Y^2\right] - M\Lambda v^\nu, \qquad Y \equiv \frac{\dot{v}}{v}, \tag{153}$$

where

$$B_{24} \equiv -\frac{2\alpha_0 m\,(2m-3)\,(2m+1)}{\Pi_4}, \qquad C_{24} \equiv -\frac{4m^2}{\Pi_4}. \tag{154}$$

The conserved momentum associated with the cyclic coordinate is

$$\Sigma_{\mathrm{B2}\cap\mathrm{B4}} = h_0 B_{24} v^m Y. \tag{155}$$

The equation obtained by varying v in the reduced Lagrangian (153), using the Euler–Lagrange equation, is

$$B_{24}\ddot{z} + 2C_{24}\dot{Y} + mC_{24}Y^2 + \frac{M\Lambda\nu}{h_0} v^{\nu-m} = 0. \tag{156}$$

The Hamiltonian energy function is

$$E_{\mathrm{B2}\cap\mathrm{B4}} = h_0 v^m \left[B_{24}\dot{z}\,Y + C_{24}\,Y^2\right] + M\Lambda v^\nu. \tag{157}$$

Imposing $E_{\mathrm{B2}\cap\mathrm{B4}} = 0$ gives

$$Y = \frac{\Sigma_{\mathrm{B2}\cap\mathrm{B4}}}{h_0 B_{24} v^m}, \qquad \dot{z} = -\frac{C_{24}\Sigma_{\mathrm{B2}\cap\mathrm{B4}}}{h_0 B_{24}^2} v^{-m} - \frac{M\Lambda}{\Sigma_{\mathrm{B2}\cap\mathrm{B4}}} v^\nu. \tag{158}$$

The constrained system requires $\Sigma_{\mathrm{B2}\cap\mathrm{B4}} \neq 0$. From (158), one finds

$$v(t) = \left[v_0^m + \frac{m\Sigma_{\mathrm{B2}\cap\mathrm{B4}}}{h_0 B_{24}}\,(t-t_0)\right]^{\frac{1}{m}}. \tag{159}$$

For $\nu + m \neq 0$, the cyclic variable is

$$z(t) = z_0 - \frac{C_{24}}{B_{24}} \ln\left(\frac{v(t)}{v_0}\right) - \frac{h_0 B_{24} M\Lambda}{\Sigma_{\mathrm{B2}\cap\mathrm{B4}}^2\,(\nu+m)} \left[v(t)^{\nu+m} - v_0^{\nu+m}\right]. \tag{160}$$

For $\nu + m = 0$, it becomes

$$z(t) = z_0 - \left[\frac{C_{24}}{B_{24}} + \frac{h_0 B_{24} M\Lambda}{\Sigma_{\mathrm{B2}\cap\mathrm{B4}}^2}\right] \ln\left(\frac{v(t)}{v_0}\right). \tag{161}$$

Therefore,

$$a(t) = \mathrm{e}^{\alpha_0 z(t)}, \qquad \xi(t) = v(t)\,\mathrm{e}^{-3\alpha_0 z(t)/m}. \tag{162}$$

3.3.3. Degenerate regular subcase

There is also the degenerate regular subcase mentioned after (59), namely

$$m = \frac{1}{2}, \qquad \mu = -2, \qquad r \neq -12. \tag{163}$$

For this subcase, the general expressions (54) give

$$\sigma = \frac{r-12}{r+12}, \qquad \ell = \frac{r}{r+12}. \tag{164}$$

An adapted coordinate system is

$$v = \xi a^2, \qquad z = \frac{1}{2\alpha_0} a^{\frac{24}{r+12}} \xi^{-\frac{r}{r+12}}. \tag{165}$$

The inverse transformation is

$$a = (2\alpha_0 z)^{1/2} v^{\frac{r}{2(r+12)}}, \qquad \xi = (2\alpha_0 z)^{-1} v^{\frac{12}{r+12}}. \tag{166}$$

The reduced Lagrangian is

$$\mathcal{L}_{\mathrm{B4d}} = \bar{A}_4 v^{\bar{s}_4} \dot{z} Y - M\Lambda v^{\nu}, \qquad Y \equiv \frac{\dot{v}}{v}, \tag{167}$$

where

$$\bar{A}_4 \equiv -\frac{\alpha_0 h_0 (r+12)}{4}, \qquad \bar{s}_4 \equiv \frac{3(r+4)}{2(r+12)}. \tag{168}$$

The conserved momentum, the v Euler–Lagrange equation, and the Hamiltonian energy function obtained from (167) are

$$\Sigma_{\mathrm{B4d}} = \bar{A}_4 v^{\bar{s}_4} Y, \tag{169}$$

$$\bar{A}_4 \ddot{z} + M\Lambda \nu v^{\nu-\bar{s}_4} = 0, \tag{170}$$

and

$$E_{\mathrm{B4d}} = \bar{A}_4 v^{\bar{s}_4} \dot{z} Y + M\Lambda v^{\nu}. \tag{171}$$

After imposing $E_{\mathrm{B4d}} = 0$, one obtains

$$Y = \frac{\Sigma_{\mathrm{B4d}}}{\bar{A}_4 v^{\bar{s}_4}}, \qquad \dot{z} = -\frac{M\Lambda}{\Sigma_{\mathrm{B4d}}} v^{\nu}. \tag{172}$$

For $\bar{s}_4 \neq 0$,

$$v(t) = \left[v_0^{\bar{s}_4} + \frac{\bar{s}_4 \Sigma_{\mathrm{B4d}}}{\bar{A}_4} (t - t_0) \right]^{\frac{1}{\bar{s}_4}}, \tag{173}$$

whereas for $\bar{s}_4 = 0$,

$$v(t) = v_0 \exp\left[\frac{\Sigma_{\mathrm{B4d}}}{\bar{A}_4} (t - t_0) \right]. \tag{174}$$

For $\nu + \bar{s}_4 \neq 0$, the cyclic coordinate is

$$z(t) = z_0 - \frac{\bar{A}_4 M\Lambda}{\Sigma_{\mathrm{B4d}}^2 (\nu + \bar{s}_4)} \left[v(t)^{\nu+\bar{s}_4} - v_0^{\nu+\bar{s}_4} \right]. \tag{175}$$

For $\nu + \bar{s}_4 = 0$, it is

$$z(t) = z_0 - \frac{\bar{A}_4 M\Lambda}{\Sigma_{\mathrm{B4d}}^2} \ln\left(\frac{v(t)}{v_0} \right). \tag{176}$$

Finally, the corresponding scale factor and field variable are

$$a(t) = (2\alpha_0 z(t))^{1/2} v(t)^{\frac{r}{2(r+12)}}, \qquad \xi(t) = (2\alpha_0 z(t))^{-1} v(t)^{\frac{12}{r+12}}. \tag{177}$$

Equations (148), (162) and (177) summarize the constrained cosmological solutions arising from the admissible sectors of Branch 4.

4. Homogeneous stability of the Noether branches

In this section we study the homogeneous stability of the exact cosmological solutions obtained from the Noether-reduced dynamics. The analysis is performed for the retained branches in the adapted variables used in the previous section. For each branch, the background solution is perturbed inside the same minisuperspace sector, and the linearized Euler–Lagrange equations are supplemented by the Hamiltonian constraint and the conserved Noether charge. Since some branches have degenerate kinetic sectors, stability is not inferred only from the kinetic matrix. Instead, the independent perturbation equations are derived explicitly. A solution is regarded as asymptotically stable only if all independent homogeneous perturbations decay at late times. This criterion concerns homogeneous evolution only and does not establish the absence of ghosts or gradient instabilities in the four-dimensional theory. In particular, the Branch 3 results remain subject to the negative-kinetic-coupling limitation discussed in section 3.2.

4.1. Branch 2

We now study the homogeneous perturbations of the generic non-overlapping sector of Branch 2. In this sector the adapted variables are given by (67), and the reduced Lagrangian is (69). The kinetic sector is non-degenerate in the generic sector because $A_2 \neq 0$ and $G_2 \neq 0$, with G_2 defined in (78). Therefore, the invariant variable v is a dynamical variable in the reduced system. It is useful to write

$$Y \equiv \frac{\dot{v}}{v}, \qquad q \equiv \ln v. \tag{178}$$

The physical perturbations follow directly from (68). For

$$v(t) = v_{\rm b}(t)\left[1 + \Delta_v(t)\right], \qquad z(t) = z_{\rm b}(t) + \delta z(t), \tag{179}$$

one obtains, to first order,

$$\frac{\delta a}{a} = \alpha_0 \delta z, \qquad \frac{\delta \xi}{\xi} = \Delta_v - \frac{3\alpha_0}{m}\delta z. \tag{180}$$

4.1.1. Zero-charge sector

We first consider the zero-charge solution (86). The background satisfies

$$\Sigma_{\rm B2} = 0, \qquad \dot{z}_{\rm b} = -\frac{B_2}{2A_2} Y_{\rm b}, \qquad Y_{\rm b}^2 = \kappa_2^2 v_{\rm b}^{\nu - m}, \tag{181}$$

where κ_2 is defined in (82). For $\nu \neq m$, the scale factor behaves as a power law,

$$a(t) \propto (t - t_*)^{P_2}, \qquad P_2 \equiv \frac{2p_2}{m - \nu}, \tag{182}$$

on a real branch where the argument of the power is positive. The corresponding effective equation-of-state parameter is

$$\omega_{\rm eff}^{(\rm B2,0)} = -1 + \frac{2}{3P_2} = -1 + \frac{m - \nu}{3p_2}. \tag{183}$$

For $\nu = m$, the zero-charge solution is exponential in time, with $a(t) \propto \exp(\varepsilon p_2 \kappa_2 t)$.

Let $\delta\Sigma_{\rm B2}$ denote the first-order perturbation of the conserved Noether charge. Linearizing the charge relation (72) around the zero-charge background gives

$$2A_2 \delta\dot{z} + B_2 \dot{\Delta}_v = \frac{\delta\Sigma_{\rm B2}}{h_0 v_{\rm b}^m}. \tag{184}$$

The linearized Hamiltonian constraint gives the independent first-order equation

$$2\dot{\Delta}_v + (m - \nu) Y_{\rm b} \Delta_v = 0. \tag{185}$$

The linearized equation obtained by varying v in (69) is

$$\ddot{\Delta}_v + m Y_{\rm b} \dot{\Delta}_v + \frac{\nu(m - \nu)}{2} Y_{\rm b}^2 \Delta_v = 0. \tag{186}$$

Equation (186) is not independent, since it follows from (185) and the background equation for $Y_{\rm b}$.

For $\nu \neq m$, equation (185) gives

$$\Delta_v(t) = C_v v_{\rm b}(t)^{-\frac{m-\nu}{2}}. \tag{187}$$

Then equation (184) yields

$$\delta z(t) = C_z - \frac{B_2}{2A_2}\Delta_v(t) + \frac{\delta\Sigma_{\rm B2}}{2A_2 h_0}\int^t v_{\rm b}(u)^{-m}\,{\rm d}u. \tag{188}$$

Using (83), the last integral can be written as

$$\int^t v_{\rm b}(u)^{-m}\,{\rm d}u = -\frac{2}{\varepsilon\kappa_2(m + \nu)} v_{\rm b}(t)^{-\frac{m+\nu}{2}}, \qquad m + \nu \neq 0, \tag{189}$$

where an additive constant has been absorbed into C_z. In the special case $m+\nu=0$, one instead obtains

$$\int^{t} v_{\rm b}(u)^{-m}\,{\rm d}u=\frac{1}{\varepsilon\kappa_2}\ln\left(\frac{v_{\rm b}(t)}{v_0}\right). \tag{190}$$

Equations (187)–(190) determine the homogeneous perturbations of the zero-charge solution. The constant C_z is a neutral shift of the cyclic coordinate and therefore corresponds to a constant normalization shift of a and ξ. The mode proportional to C_v decays only on future branches for which

$$v_{\rm b}(t)^{\frac{m-\nu}{2}}\to\infty. \tag{191}$$

If perturbations with $\delta\Sigma_{\rm B2}\neq 0$ are allowed, the charge-induced contribution decays only when

$$v_{\rm b}(t)^{\frac{m+\nu}{2}}\to\infty,\qquad m+\nu\neq 0. \tag{192}$$

For $m+\nu=0$, the charge-induced contribution is logarithmic and does not decay. Therefore, the zero-charge Branch 2 solution is not an asymptotic attractor under the strict criterion that all homogeneous perturbations must decay, because of the neutral normalization mode. After fixing this normalization, it is conditionally stable inside the fixed-charge sector $\delta\Sigma_{\rm B2}=0$ if (191) holds. If perturbations away from the zero-charge hypersurface are included, the additional condition (192) is also required.

For the special zero-charge case $\nu=m$, equation (185) reduces to

$$\dot{\Delta}_v=0. \tag{193}$$

Thus Δ_v contains a constant mode. Consequently, the zero-charge exponential solution with $\nu=m$ is not asymptotically stable in the full homogeneous minisuperspace.

4.1.2. Constant-invariant sector

We next consider the constant-invariant exponential solution (90). This solution exists for $\nu=-m$. We write

$$v(t)=v_0\left[1+\Delta_v(t)\right],\qquad z(t)=z_{\rm b}(t)+\delta z(t),\qquad z_{\rm b}(t)=z_0+Z_2(t-t_0),\qquad Z_2\equiv\frac{\lambda}{\alpha_0}. \tag{194}$$

Linearizing the Hamiltonian constraint gives

$$2A_2\delta\dot{z}+B_2\dot{\Delta}_v+2mA_2Z_2\Delta_v=0. \tag{195}$$

The linearized charge relation has the same left-hand side as (195). Hence the Hamiltonian constraint enforces

$$\delta\Sigma_{\rm B2}=0 \tag{196}$$

at first order. The remaining independent linearized equation is obtained from the v-equation of (69) and gives

$$\ddot{\Delta}_v=0. \tag{197}$$

Therefore,

$$\Delta_v(t)=C_v^{(0)}+C_v^{(1)}(t-t_0). \tag{198}$$

Substitution into (195) gives

$$\delta z(t)=C_z-\frac{B_2}{2A_2}C_v^{(1)}(t-t_0)-mZ_2C_v^{(0)}(t-t_0)-\frac{mZ_2}{2}C_v^{(1)}(t-t_0)^2, \tag{199}$$

where a constant term has been absorbed into C_z.

Equations (198) and (199) show that the constant-invariant exponential solution of Branch 2 has non-decaying and growing homogeneous modes. A constant displacement in v_0 changes the member of the exponential family and produces a term linear in time in the cyclic perturbation. A perturbation with $C_v^{(1)}\neq 0$ gives a quadratic contribution to δz. Therefore, the constant-invariant Branch 2 exponential solution is not asymptotically stable in the full homogeneous minisuperspace. After fixing the invariant direction and the normalization mode, the remaining perturbation can be removed only by setting $C_v^{(0)}=C_v^{(1)}=0$, so the solution is not an attractor.

4.2. Branch 3

4.2.1. Generic constant-invariant scaling sector

We next consider the generic non-overlapping sector of Branch 3. The reduced Lagrangian is given by (103). In this branch, the invariant variable ν appears without a time derivative. Therefore, the kinetic sector is degenerate and the equation obtained by varying ν in (103) acts as a constraint. The constant-invariant solution (115) exists after imposing (109). We define

$$D_3 \equiv m\mu + 3, \qquad p_3 \equiv \frac{2}{D_3}. \tag{200}$$

Thus the generic Branch 3 scaling solution has $a(t) \propto (t - t_*)^{p_3}$, with $D_3 \neq 0$ and the regularity conditions stated in (96). On an expanding future branch with $t > t_*$, the corresponding effective equation-of-state parameter is

$$\omega_{\rm eff}^{(\rm B3)} = -1 + \frac{2}{3p_3} = \frac{m\mu}{3}. \tag{201}$$

Hence the Branch 3 power-law background is accelerating when $0 < D_3 < 2$.

We perturb the constant-invariant solution as

$$\tau \equiv t - t_*, \qquad \nu(t) = \nu_0\,[1 + \Delta_\nu(t)], \qquad z(t) = z_{\rm b}(t)\,[1 + \Delta_z(t)], \qquad z_{\rm b}(t) = z_1\tau, \tag{202}$$

where $|\Delta_\nu| \ll 1$ and $|\Delta_z| \ll 1$. Linearizing the Hamiltonian constraint (107), after using (109), gives

$$\tau\dot{\Delta}_z + \Delta_z + s_3\Delta_\nu = 0. \tag{203}$$

The same relation follows from the linearized conserved charge (110). The linearized equation obtained by varying ν in (103) does not provide an independent condition. For $s_3 \neq 0$ it reduces to (203) with the opposite sign, while for $s_3 = 0$ it is identically satisfied. Therefore, the complete first-order perturbation equation in the generic Branch 3 sector is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau\Delta_z) = -s_3\Delta_\nu. \tag{204}$$

Its solution is

$$\Delta_z(\tau) = \frac{C_z}{\tau} - \frac{s_3}{\tau}\int^{\tau}\Delta_\nu(u)\,\mathrm{d}u, \tag{205}$$

where C_z is an integration constant. No differential equation is obtained for Δ_ν, which reflects the degeneracy of (103).

The perturbations of the physical variables follow from (102). To first order,

$$\frac{\delta a}{a} = \frac{2}{D_3}\Delta_z - \frac{m}{\Delta_3}\Delta_\nu, \qquad \frac{\delta\xi}{\xi} = \frac{2\mu}{D_3}\Delta_z - \frac{2(m-1)}{\Delta_3}\Delta_\nu. \tag{206}$$

Using (205), these become

$$\frac{\delta a}{a} = \frac{2C_z}{D_3\tau} - \frac{2s_3}{D_3\tau}\int^{\tau}\Delta_\nu(u)\,\mathrm{d}u - \frac{m}{\Delta_3}\Delta_\nu, \tag{207}$$

and

$$\frac{\delta\xi}{\xi} = \frac{2\mu C_z}{D_3\tau} - \frac{2\mu s_3}{D_3\tau}\int^{\tau}\Delta_\nu(u)\,\mathrm{d}u - \frac{2(m-1)}{\Delta_3}\Delta_\nu. \tag{208}$$

If the perturbations are restricted to the constant-invariant subspace, $\Delta_\nu = 0$, then

$$\frac{\delta a}{a} = \frac{2C_z}{D_3\tau}, \qquad \frac{\delta\xi}{\xi} = \frac{2\mu C_z}{D_3\tau}. \tag{209}$$

These perturbations decay as τ^{-1} on the future branch. Therefore, the Branch 3 power-law solution is stable only with respect to perturbations that preserve the Noether invariant. In the full homogeneous perturbation space, Δ_ν is not dynamically fixed. Consequently, a constant Δ_ν gives a non-decaying displacement along the family of constant-invariant solutions, while a time-dependent Δ_ν is not suppressed by the reduced equations. Thus the generic Branch 3 power-law solution is not an asymptotic attractor in the full homogeneous minisuperspace.

4.2.2. Intersection with Branch 2

We also consider the overlapping sector B2 ∩ B3, whose reduced Lagrangian is (117). The exponential solution (124) is obtained for $\nu = -m$. We perturb it as

$$v(t) = v_0\,[1 + \Delta_v(t)]\,, \qquad z(t) = z_{\rm b}(t) + \delta z(t)\,, \qquad z_{\rm b}(t) = z_0 + \frac{\lambda}{\alpha_0}\,(t - t_0)\,. \tag{210}$$

The velocity combination $\mathcal{Q}_3$, defined in (118), has the background value

$$\mathcal{Q}_{3\rm b} = (2m+1)\,\lambda. \tag{211}$$

Linearizing the Hamiltonian constraint and the conserved charge gives

$$\alpha_0\,(2m+1)\,\delta\dot{z} - m\dot{\Delta}_v + m\,(2m+1)\,\lambda\Delta_v = 0. \tag{212}$$

Equivalently,

$$\alpha_0\delta z(t) = C_z + \frac{m}{2m+1}\Delta_v(t) - m\lambda\int^t \Delta_v(u)\,{\rm d}u. \tag{213}$$

The inverse map in this sector is the same as (68). Therefore,

$$\frac{\delta a}{a} = C_z + \frac{m}{2m+1}\Delta_v - m\lambda\int^t \Delta_v(u)\,{\rm d}u, \tag{214}$$

and

$$\frac{\delta\xi}{\xi} = -\frac{3}{m}C_z + \frac{2\,(m-1)}{2m+1}\Delta_v + 3\lambda\int^t \Delta_v(u)\,{\rm d}u. \tag{215}$$

The linearized v-equation obtained from (117) is not independent, since it follows from (212) and the background equations. Thus Δ_v remains arbitrary in the overlapping sector. For $\Delta_v = 0$, the remaining perturbation is a constant normalization shift and does not decay. For a nonzero constant Δ_v, the integral terms in (214) and (215) grow linearly in time. Therefore, the overlapping Branch 3 exponential solution is not asymptotically stable in the full homogeneous minisuperspace. It is at most neutrally stable after the invariant direction and the normalization mode have been fixed.

4.3. Branch 4

4.3.1. Generic constrained sector

We now analyze the homogeneous perturbations of Branch 4. In the generic regular sector, the reduced Lagrangian is (136). Although the kinetic term contains only the mixed product $\dot{z}\dot{v}/v$, the kinetic sector in the variables $(z, \ln v)$ is non-degenerate when $A_4 \neq 0$. In contrast with Branch 3, the constrained dynamics fixes both $v(t)$ and $z(t)$ through the first-order system (142). The constant-invariant sector is not compatible with the Hamiltonian constraint for $M\Lambda \neq 0$.

We perturb the generic Branch 4 solution as

$$v(t) = v_{\rm b}(t)\,[1 + \Delta_v(t)]\,, \qquad z(t) = z_{\rm b}(t) + \delta z(t)\,, \tag{216}$$

where $v_{\rm b}$ and $z_{\rm b}$ satisfy (142). We also allow a first-order perturbation of the conserved Noether charge,

$$\chi_4 \equiv \frac{\delta\Sigma_{\rm B4}}{\Sigma_{\rm B4}}, \qquad \Sigma_{\rm B4} \neq 0. \tag{217}$$

Since

$$Y = \frac{\dot{v}}{v}, \tag{218}$$

one has $Y = Y_{\rm b} + \dot{\Delta}_v$ at first order. Linearizing the conserved charge (138) gives

$$\dot{\Delta}_v + s_4 Y_{\rm b}\Delta_v = \chi_4 Y_{\rm b}. \tag{219}$$

The linearized Hamiltonian constraint gives

$$\delta\dot{z} = \dot{z}_{\rm b}\,(\nu\Delta_v - \chi_4)\,. \tag{220}$$

The linearized equation obtained by varying v in (136) follows from (219), (220), and the background equations; hence it does not add an independent perturbation equation.

For $s_4 \neq 0$, equation (219) gives

$$\Delta_v(t) = \frac{\chi_4}{s_4} + C_v v_{\rm b}(t)^{-s_4}. \tag{221}$$

Using (220), the corresponding perturbation of the cyclic coordinate is

$$\delta z(t) = C_z + \left(\frac{\nu}{s_4} - 1\right)\chi_4 z_{\rm b}(t) - \frac{A_4 M\Lambda}{\Sigma_{\rm B4}^2} C_v v_{\rm b}(t)^{\nu}, \tag{222}$$

where additive constants have been absorbed into C_z. For $s_4 = 0$, equation (219) instead gives

$$\Delta_v(t) = C_v + \chi_4 \ln\left(\frac{v_{\rm b}(t)}{v_0}\right). \tag{223}$$

Thus, even in the fixed-charge sector $\chi_4 = 0$, the case $s_4 = 0$ contains a non-decaying invariant perturbation.

The physical perturbations follow from (135). For compactness we define

$$D_4 \equiv m\mu + 3, \qquad \gamma_4 \equiv \frac{m\mu + 1}{(\mu+2)\Delta_4}, \qquad \eta_4 \equiv \frac{\Gamma_4}{(\mu+2)\Delta_4}. \tag{224}$$

Then, to first order,

$$\frac{\delta a}{a} = \frac{1}{D_4}\frac{\delta z}{z_{\rm b}} - \gamma_4 \Delta_v, \qquad \frac{\delta\xi}{\xi} = \frac{\mu}{D_4}\frac{\delta z}{z_{\rm b}} + \eta_4 \Delta_v. \tag{225}$$

Equations (221) and (222) show that a perturbation with $\chi_4 \neq 0$ changes the conserved charge and produces a non-decaying displacement in the reduced variables. Therefore, the generic Branch 4 solution is not an asymptotic attractor in the full homogeneous minisuperspace when charge perturbations are included.

If the charge is fixed, $\chi_4 = 0$, and $s_4 \neq 0$, the remaining invariant perturbation is proportional to $v_{\rm b}^{-s_4}$. In this fixed-charge sector, the perturbations decay only if

$$v_{\rm b}(t)^{s_4} \to \infty, \qquad z_{\rm b}(t) \to \infty, \qquad \frac{v_{\rm b}(t)^{\nu}}{z_{\rm b}(t)} \to 0 \tag{226}$$

on the future branch. These conditions ensure that both Δ_v and $\delta z/z_{\rm b}$ vanish asymptotically. Hence Branch 4 can be conditionally stable in the fixed-charge sector, but it is not stable under arbitrary homogeneous perturbations of the conserved charge.

4.3.2. Intersection with Branch 2

The limiting sector B2 ∩ B4 is described by the reduced Lagrangian (153). We perturb the solution (162) as

$$v(t) = v_{\rm b}(t)[1 + \Delta_v(t)], \qquad z(t) = z_{\rm b}(t) + \delta z(t), \qquad \chi_{24} \equiv \frac{\delta\Sigma_{\rm B2\cap B4}}{\Sigma_{\rm B2\cap B4}}. \tag{227}$$

Linearizing the conserved charge (155) gives

$$\dot{\Delta}_v + m Y_{\rm b} \Delta_v = \chi_{24} Y_{\rm b}. \tag{228}$$

Since $m \neq 0$ in this sector, the solution is

$$\Delta_v(t) = \frac{\chi_{24}}{m} + C_v v_{\rm b}(t)^{-m}. \tag{229}$$

The linearized Hamiltonian constraint gives

$$\delta\dot{z} = -\frac{C_{24}\Sigma_{\rm B2\cap B4}}{h_0 B_{24}^2} v_{\rm b}^{-m}(\chi_{24} - m\Delta_v) - \frac{M\Lambda}{\Sigma_{\rm B2\cap B4}} v_{\rm b}^{\nu}(-\chi_{24} + \nu\Delta_v). \tag{230}$$

After substituting (229), one obtains

$$\delta z(t) = C_z - \frac{C_{24}}{B_{24}} C_v \nu_{\rm b}(t)^{-m} - \frac{h_0 B_{24} M\Lambda}{\Sigma^2_{\rm B2\cap B4}} C_v \nu_{\rm b}(t)^{\nu} + \left(\frac{\nu}{m} - 1\right) \chi_{24} \mathcal{I}_{24}(t), \tag{231}$$

where additive constants have again been absorbed into C_z, and

$$\mathcal{I}_{24}(t) = -\frac{h_0 B_{24} M\Lambda}{\Sigma^2_{\rm B2\cap B4}(\nu+m)} \nu_{\rm b}(t)^{\nu+m}, \qquad \nu+m \neq 0, \tag{232}$$

while for $\nu + m = 0$,

$$\mathcal{I}_{24}(t) = -\frac{h_0 B_{24} M\Lambda}{\Sigma^2_{\rm B2\cap B4}} \ln\left(\frac{\nu_{\rm b}(t)}{\nu_0}\right). \tag{233}$$

The physical perturbations follow from (152),

$$\frac{\delta a}{a} = \alpha_0 \delta z, \qquad \frac{\delta \xi}{\xi} = \Delta_\nu - \frac{3\alpha_0}{m} \delta z. \tag{234}$$

The charge perturbation χ_{24} produces a non-decaying contribution to Δ_ν, and for $\nu + m = 0$ it also gives a logarithmic contribution to δz. Therefore, the B2 ∩ B4 solution is not an asymptotic attractor in the full homogeneous minisuperspace. In the fixed-charge sector, after removing the constant normalization mode, the remaining perturbations can decay only if the time-dependent terms in (229) and (231) vanish asymptotically.

4.3.3. Degenerate regular subcase

Finally, the degenerate regular subcase of Branch 4, described by (167), has the same perturbation structure as the generic Branch 4 sector after the replacements

$$A_4 \to \bar{A}_4, \qquad s_4 \to \bar{s}_4, \qquad \Sigma_{\rm B4} \to \Sigma_{\rm B4d}. \tag{235}$$

Thus, for $\bar{s}_4 \neq 0$,

$$\Delta_\nu(t) = \frac{\chi_{4d}}{\bar{s}_4} + C_v \nu_{\rm b}(t)^{-\bar{s}_4}, \qquad \chi_{4d} \equiv \frac{\delta\Sigma_{\rm B4d}}{\Sigma_{\rm B4d}}, \tag{236}$$

and

$$\delta z(t) = C_z + \left(\frac{\nu}{\bar{s}_4} - 1\right) \chi_{4d} z_{\rm b}(t) - \frac{\bar{A}_4 M\Lambda}{\Sigma^2_{\rm B4d}} C_v \nu_{\rm b}(t)^{\nu}. \tag{237}$$

For $\bar{s}_4 = 0$, the invariant perturbation contains a constant mode in the fixed-charge sector and a logarithmic mode if $\chi_{4d} \neq 0$. The physical perturbations are obtained from (166),

$$\frac{\delta a}{a} = \frac{1}{2}\frac{\delta z}{z_{\rm b}} + \frac{r}{2(r+12)} \Delta_\nu, \qquad \frac{\delta \xi}{\xi} = -\frac{\delta z}{z_{\rm b}} + \frac{12}{r+12} \Delta_\nu. \tag{238}$$

Therefore, the degenerate regular Branch 4 subcase has the same qualitative stability property as the generic Branch 4 sector: perturbations of the conserved charge generate non-decaying modes, whereas conditional decay is possible only in a fixed-charge sector and only for future branches satisfying the corresponding decay conditions.

4.4. Summary of the homogeneous stability analysis

The homogeneous stability analysis of the retained sectors shows that Branches 2–4 have different reduced kinetic structures, and this affects the interpretation of their perturbations. Branch 3 has a degenerate kinetic sector in its generic power-law sector. In this case, the Noether invariant ν is not dynamically fixed by the reduced equations. Consequently, the power-law solution has a decaying mode proportional to τ^{-1} only after the invariant direction is fixed. In the full homogeneous minisuperspace, arbitrary perturbations in ν are not suppressed, and the corresponding solution is not an asymptotic attractor. The overlapping exponential sector B2 ∩ B3 contains a neutral normalization mode and linearly growing contributions for nonzero invariant displacements.

Branches 2 and 4 differ from Branch 3 because their generic reduced dynamics fixes the invariant variable through first-order equations. In Branch 2, the zero-charge solution may have decaying invariant

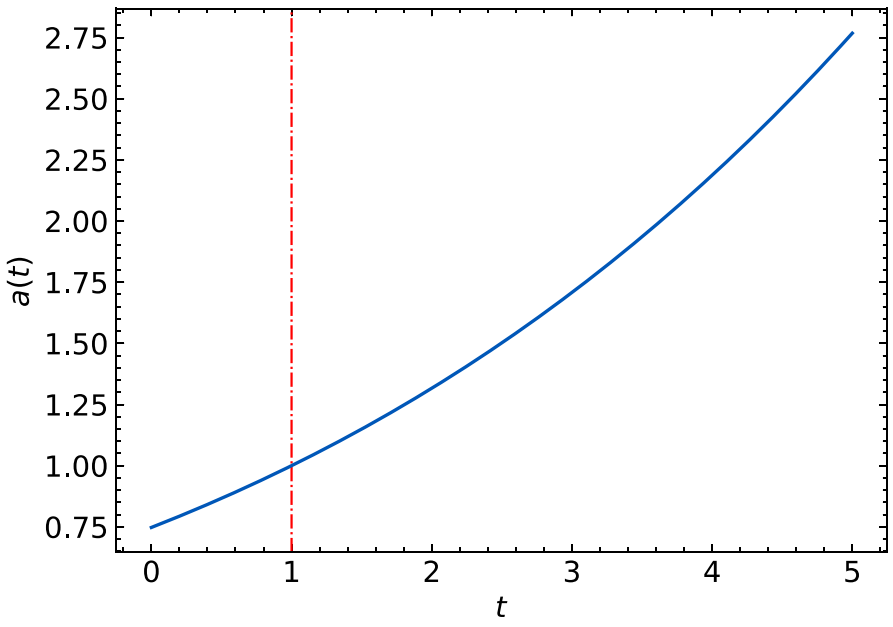


Figure 1. Evolution of the normalized scale factor $a(t)$ for the zero-charge sector of Branch 2 with $m = 2$, $r = 1$, $\nu = 9/5$, $\alpha_0 = 1$, $h_0 = 1$, $M = 1$, $\Lambda = 1$, $\nu_0 = 1$, and $\xi_0 = 1$. The scale factor is normalized by $a(t_0) = 1$, with $t_0 = 1$.

perturbations under the conditions stated in (191) and (192), but it still contains a neutral normalization mode and is not an attractor under the strict criterion that all homogeneous perturbations decay. The constant-invariant exponential solution of Branch 2 contains non-decaying and growing modes. In Branch 4, the generic solution can be conditionally stable only in a fixed-charge sector and only when the asymptotic conditions (226) hold. Perturbations that change the conserved Noether charge produce non-decaying displacements. Therefore, none of the retained branches provides an unconditional asymptotic attractor in the full homogeneous minisuperspace. The stable behavior found in this section is conditional and is obtained only after fixing the invariant direction, the conserved charge, or the normalization mode, depending on the branch.

5. Numerical illustrations of representative Noether solutions

In this subsection, we present numerical illustrations of representative exact solutions obtained from the Noether-reduced dynamics. The purpose of these plots is not to constrain the model observationally, but to show the qualitative background evolution allowed by the analytical branches. The time variable is taken to be dimensionless, and the reference epoch is fixed at $t_0 = 1$. In all plots the scale factor is normalized by $a(t_0) = 1$. The vertical red dot-dashed line denotes the reference time. Since the symbol q has already been used for the logarithmic invariant variable in the stability analysis, the deceleration parameter is denoted here by $Q_{\rm dec}$.

We first consider the zero-charge sector of Branch 2. The corresponding solution is obtained from equations (83)–(86). For the numerical example, we choose $m = 2$, $r = 1$, $\alpha_0 = 1$, $h_0 = 1$, $M = 1$, $\Lambda = 1$, $\nu = 9/5$, $\nu_0 = 1$, $\xi_0 = 1$, and the expanding sign of the solution. This choice gives $I(\xi) > 0$ on the branch $\xi > 0$, and the reduced kinetic coefficient is such that the parameter κ_2 is real. The scale factor grows monotonically and the Hubble parameter decreases slowly, which corresponds to an accelerated power-law expansion rather than an exact de Sitter phase (see figures 1 and 2). The change of the invariant $\xi(t)$ over time is shown in figure 3, where it remains positive over time and evolves monotonically on the plotted branch. In this solution, as expected from the power-law form of the scale factor, the deceleration parameter and the effective equation-of-state parameter remain constant over the plotted interval, depending on the parameters m, ν, and r.

As a second example, we consider the generic regular sector of Branch 4. The solution is obtained from equations (143)–(148). We take $m = 1/2$, $\mu = 2$, $\nu = -1$, $\alpha_0 = 1$, $h_0 = 1$, $M = 1$, $\Lambda = 1$, $\Sigma_{\rm B4} = -2$, $\nu_0 = 1$, and $z_0 = 8$. For these values, the reconstructed kinetic ratio is $r = 12$, so that the Yang–Mills

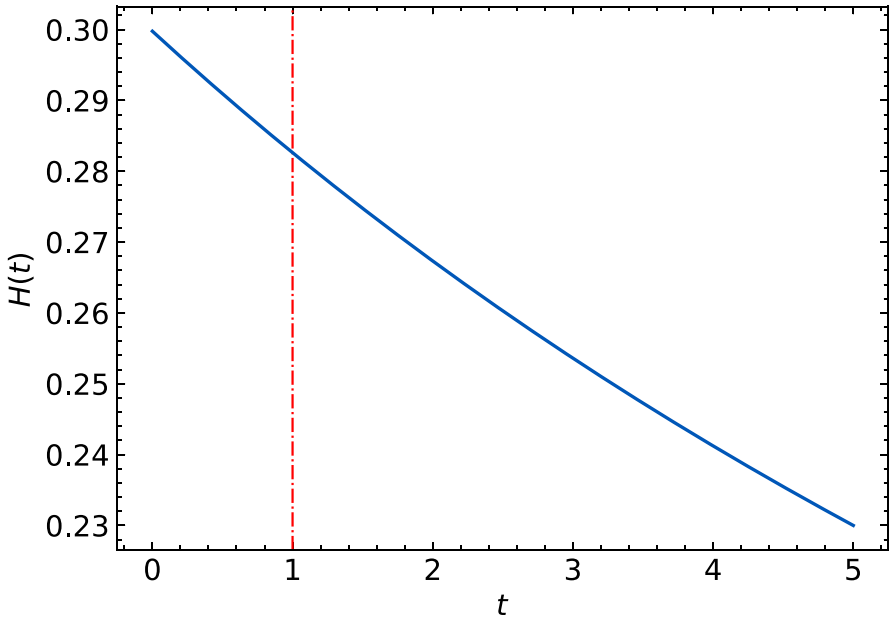


Figure 2. Evolution of the Hubble parameter $H(t)$ for the same Branch 2 parameter values as in figure 1. The decreasing behavior of $H(t)$ shows that the solution is not exactly de Sitter, although it remains in an accelerated regime.

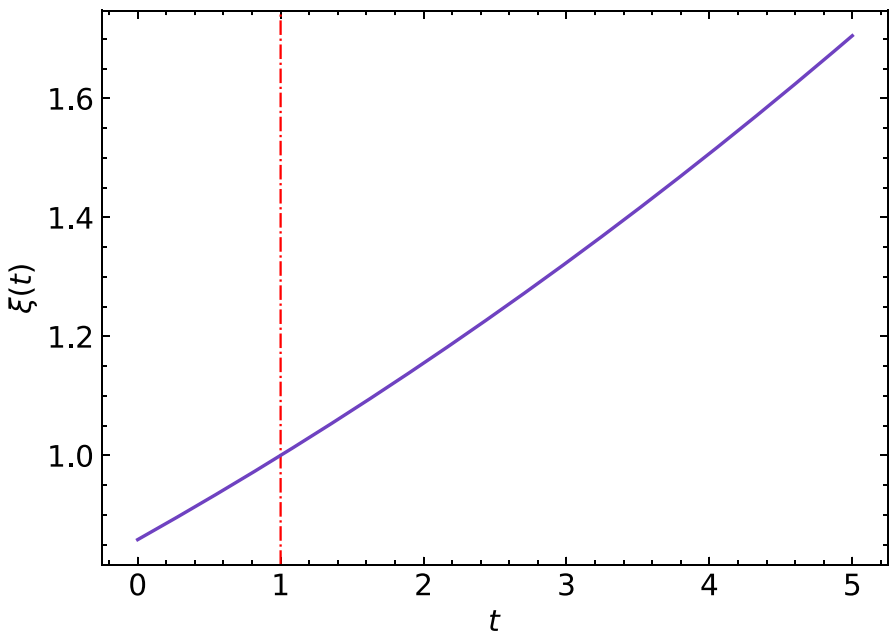


Figure 3. Evolution of the Yang–Mills invariant $\xi(t)$ for the same Branch 2 parameter values as in figure 1. The invariant remains positive and evolves monotonically on the plotted branch.

kinetic coupling is positive on the branch $\xi > 0$. This case gives a dynamical invariant and a time-dependent effective equation of state. The evolution of cosmological parameters over time is shown in the figures 4–7. The scale factor grows monotonically, while $Q_{\rm dec}$ changes sign close to the reference epoch. Hence the solution illustrates a transition from a decelerated stage to an accelerated stage within the same Noether branch.

The two numerical examples display different types of Noether-symmetric cosmological evolution. In the zero-charge Branch 2 example, the background behaves as an accelerated power-law solution

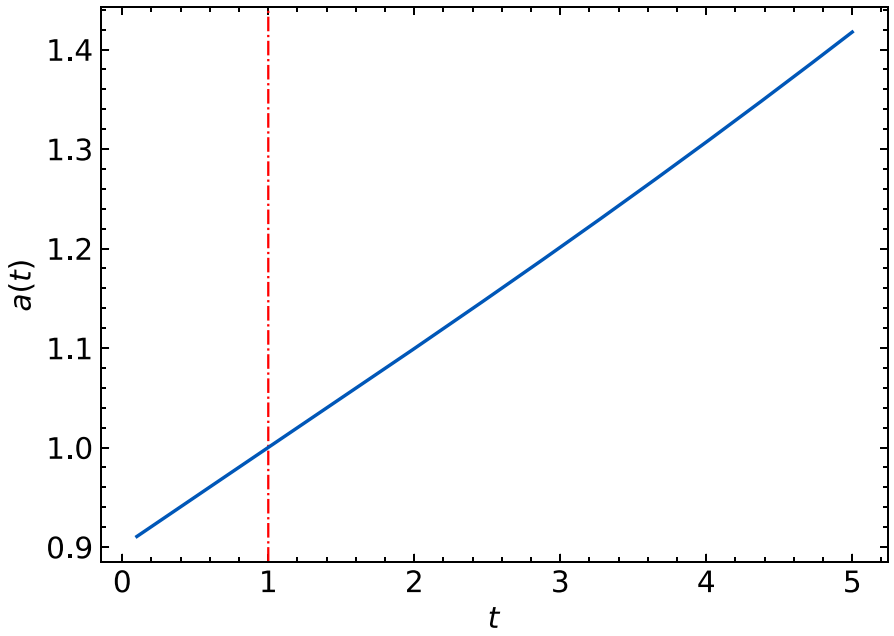


Figure 4. Evolution of the normalized scale factor $a(t)$ for the generic regular sector of Branch 4 with $m = 1/2$, $\mu = 2$, $\nu = -1$, $\alpha_0 = 1$, $h_0 = 1$, $M = 1$, $\Lambda = 1$, $\Sigma_{B4} = -2$, $v_0 = 1$, and $z_0 = 8$. The scale factor is normalized by $a(t_0) = 1$, with $t_0 = 1$.

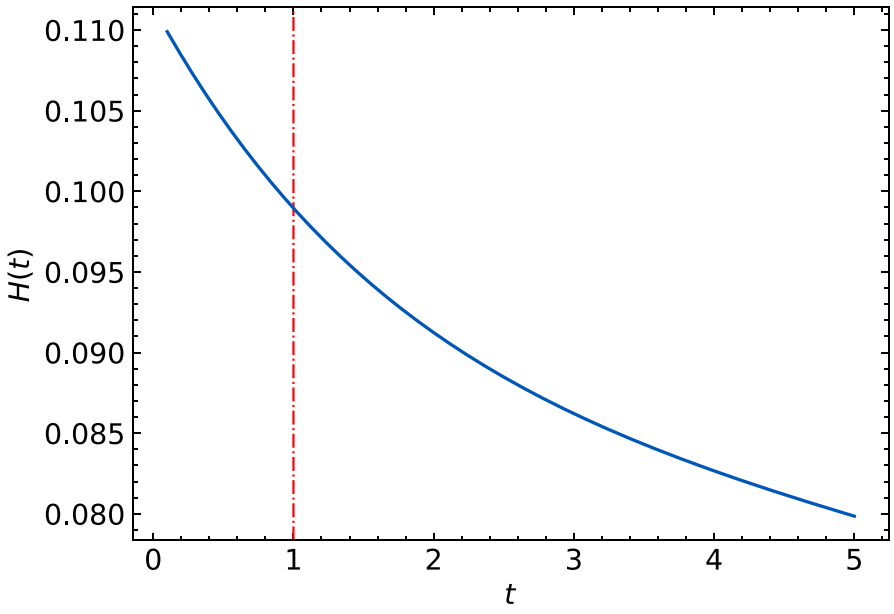


Figure 5. Evolution of the Hubble parameter $H(t)$ for the same Branch 4 parameter values as in figure 4. The Hubble parameter remains positive and decreases smoothly during the plotted evolution.

with constant $Q_{\rm dec}$ and constant $\omega_{\rm eff}$. In the Branch 4 example, the invariant $\xi(t)$ evolves with time and the effective equation of state changes during the expansion. This shows that accelerated expansion can arise either from a simple reduced power-law dynamics or from a more dynamical branch in which the Noether invariant and the cyclic coordinate both affect the background evolution.

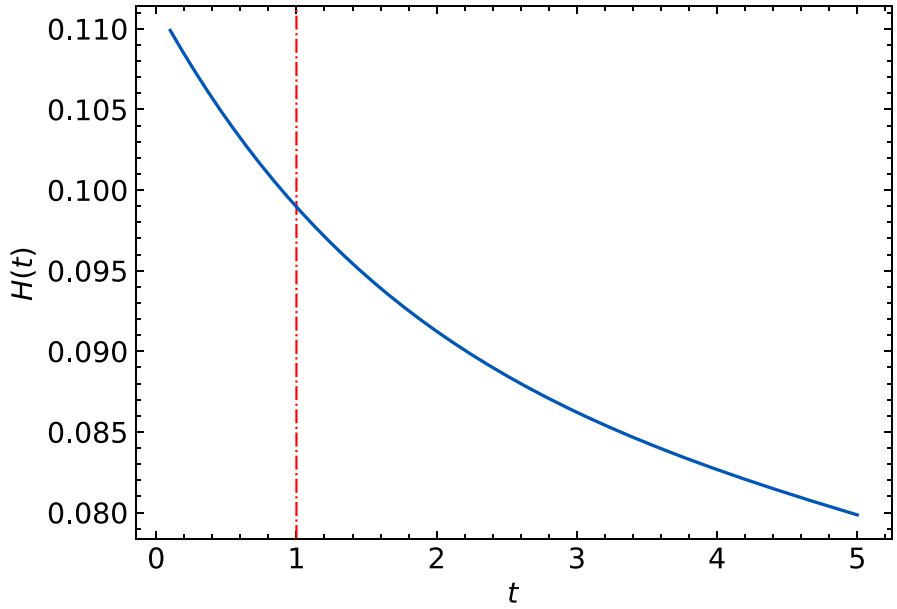


Figure 6. Evolution of the Yang–Mills invariant $\xi(t)$ for the same Branch 4 parameter values as in figure 4. The invariant remains dynamical and decreases gradually on this branch.

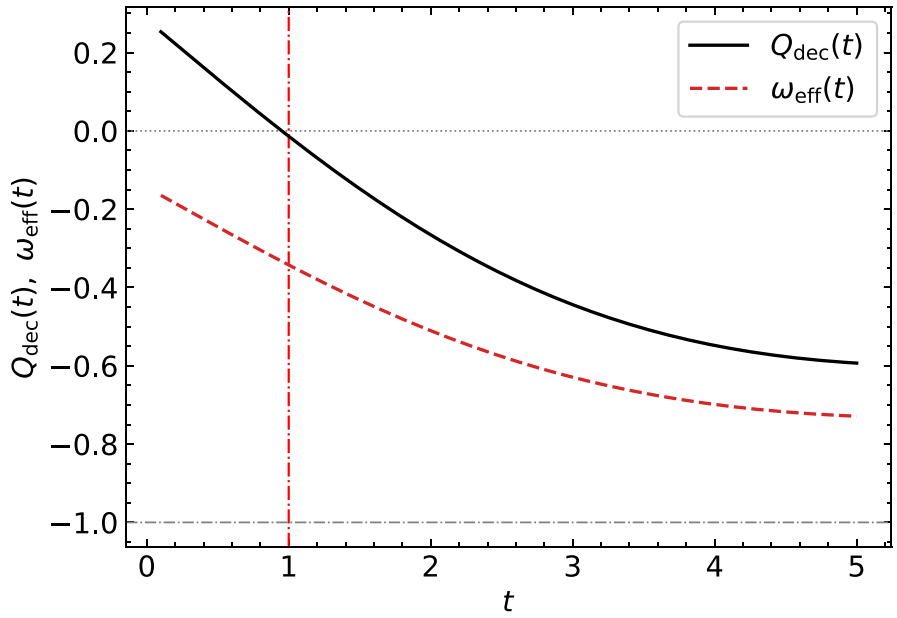


Figure 7. Evolution of $Q_{dec}(t)$ and $\omega_{eff}(t)$ for the same Branch 4 parameter values as in figure 4. The sign change of Q_{dec} shows the transition to accelerated expansion, while ω_{eff} moves into the negative-pressure regime without becoming exactly constant.

6. Conclusions

In this paper, we investigate an interacting Yang–Mills quintessence cosmological model in a spatially flat FLRW background, in which the Yang–Mills sector is described by a cosmic-triad configuration. Using this configuration, the scalar invariant ξ is defined. Starting from the action, we write the point-like Lagrangian in terms of the scale factor a and the field ξ. From this Lagrangian, we obtain the equations of motion and calculate the equation-of-state parameter for the model. The results show that this model can describe the accelerated expansion of the Universe at late times. Next, to obtain exact solutions of the proposed model, we use the Noether symmetry method. Within this approach, and by assuming power-law forms for the potential and the coupling functions, we determine the free functions of the

model. We then obtain exact solutions in four branches. The first branch, because it removes the curvature coupling, is considered only as an algebraic branch and is not discussed further in the paper, whereas the second, third, and fourth branches are investigated to construct cosmological solutions. In branch 2, power-law and exponential solutions are obtained in the zero-charge and constant-invariant sectors. In branch 3, because of the degeneracy of the kinetic sector, the dynamics does not fully determine the invariant variable. In branch 4, both the invariant and the cyclic coordinate are determined by first-order equations, and as a result, more dynamical solutions are obtained.

The paper also addresses the stability analysis and shows that the exact solutions obtained are generally not unconditional asymptotic attractors. In branch 2, stability is obtained only in specific sectors and under asymptotic conditions, while neutral or growing modes also exist. In branch 3, stability can be interpreted only after the direction of the invariant is fixed. In branch 4, conditional stability is also obtained only in a fixed-charge sector, while perturbations of the conserved charge usually produce non-decaying modes. Numerical examples also show that the second branch can exhibit accelerated power-law expansion with a decreasing H, while the fourth branch shows a transition to the accelerated phase. Therefore, we conclude that the proposed interacting Yang–Mills model provides a framework for constructing exact accelerated background solutions. These results do not establish full perturbative or observational viability. In particular, Branch 3 is retained only as a formal classical sector because of its negative Yang–Mills kinetic coupling. A complete inhomogeneous perturbation analysis and observational parameter estimation, including MCMC analyses, are left for future work.

Data availability statement

No new data were created or analysed in this study.

Author contributions

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