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NEW NTEGRAL INEQUALITIES FOR s-CONVEX FUNCTION IN THE THIRD SENSE

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AUTHORS' CONTRIBUTIONS

This study was carried out in collaboration between both authors. Author SK designed the study, made theorems and proofs. Author BU wrote the first draft of the manuscript and directed the literature research. The authors created the conclusion part of the study together. Both authors have read and approved the final article.

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ABSTRACT

In this paper, some new integral inequalities are obtained for functions whose n-th derivative is s-convex function in the third sense with the help of an integral identity. Then, by putting special conditions in these integral inequalities, new integral inequalities for classical convex functions are obtained.

Keywords: Convex function; s-convex functions in the third sense; Hölder integral inequality; power mean integral inequality.

1 INTRODUCTION

The classical definition of convex functions on a convex subset U of a vector space X is the statement that $f : U \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in U$ and $\lambda \in [0,1]$. If this inequality reverse, then f is said to be concave on U. In the definition of convexity, new classes of abstract convex functions can be produced with the conditions imposed on the coefficients [1–10]. Some of these classes are as follows;

Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a s-convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex in the first sense if

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y)$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$ [11]. Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y)$$

holds for all $x, y \in U$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$ [12].

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s-Convex functions in the third sense were introduced to the literature by Kemali et al. [8]. For this new convexity class, Hermite-Hadamard type inequalities and some integral inequalities have been studied [10, 13].The functions have been stated as the following,

Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a s-convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex function in the third sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \tag{1.1}$$

is satisfied for all $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$.

The inequality (1.1) is equivalent to the following inequalities:

$$f(\lambda^{\frac{1}{s}} x + (1-\lambda)^{\frac{1}{s}} y) \leq \lambda^{\frac{1}{s^2}} f(x) + (1-\lambda)^{\frac{1}{s^2}} f(y)$$

or

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^{\frac{1}{s}} f(x) + (1-\lambda^s)^{\frac{1}{s^2}} f(y)$$

where $\lambda \in [0,1]$ and $x, y \in U$.

In this paper, $U \subseteq \mathbb{R}^n$ will be taken as a s-convex set. The classes of s-convex functions in first, second and third senses are denoted by K_s^1, K_s^2 and K_s^3 respectively. It can be easily seen that in the case $s = 1$, each type of s-convexity is reduced to the ordinary convexity of functions. There is a large number of studies on s-convex functions and their properties, relevant inequalities mainly including Hermite-Hadamard type inequalities (see [11, 14–19] and the references therein).

Lemma 1.1. *[8] $I \subseteq \mathbb{R}$ is a convex set and also a s-convex set.*

Sarikaya *et* al. proved the following Lemma in [18] and some new integral inequalities for convex and concave functions were establish by this useful lemma. In this study, In this study, the same Lemma will be used to get the main results.

Lemma 1.2. *[18] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable function on I, $a, b \in I^o$ with $a < b$. If $f^{(n)} \in L[a,b]$, then we have the identity,*

$$\sum_{k=0}^{n-1} (-1)^k \left[\frac{f^{(k)}(b) b^{k+1} - f^{(k)}(a) a^{k+1}}{(k+1)!} \right] - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x)dx$$

where an empty sum is understood to be nil.

Throughout this paper, the following notations and conventions are used.

Let I and I^o be the subset of $\mathbb{R}$ and interior of I, respectively and

$$A(a,b) = \frac{a+b}{2}, \; L_p(a,b) = \begin{cases} a, & \text{if } a = b \\ \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)} \right)^{1/p}, & \text{else} \end{cases}$$

be Arithmetic mean and Stolarsky mean (Generalized Logarithmic mean) for positive numbers a, b, p with $a \neq b$, respectively.

2 MAIN RESULTS

In this section, some new integral inequalities for functions whose absolute value of n-th derivative are s-convex functions in the third sense are given.

Theorem 1. For $n \in \mathbb{N}$, let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be $n-$times differentiable function on I^o and let $a, b \in I^o$ with $a < b$. If $\left|f^{(n)}\right|^q$ for $q > 1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q \in L[a,b]$ for some fixed $s \in [0,1)$, then the following inequality holds;

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \leq \frac{1}{n!}(b-a)L_{np}^n(a,b)\left[\frac{s\left[\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right]}{s+1}\right]^{1/q}. \quad (2.1)$$

Proof. Let $x = \frac{x-a}{b-a}b + \frac{b-x}{b-a}a$. s-Convexity of $\left|f^{(n)}\right|^q$, we get

$$\left|f^{(n)}(x)\right|^q = \left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q \leq \left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q + \left[1-\left(\frac{x-a}{b-a}\right)\right]^{1/s}\left|f^{(n)}(a)\right|^q.$$

From Lemma (1.2) and since $\left|f^{(n)}\right|^q$ is a s-convex function in the third sense and using the well known Hölder integral inequality, we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \\
\leq\ & \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
\leq\ & \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\int_a^b\left|f^{(n)}(x)\right|^q dx\right]^{1/q} \\
\leq\ & \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\int_a^b\left[\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q+\left(1-\left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right]^{1/q} \\
=\ & \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\int_a^b(x-a)^{1/s}dx+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b\left[(b-a)^{1/s}-(x-a)^{1/s}\right]dx\right]^{1/q} \\
=\ & \frac{1}{n!}\left[\left.\frac{x^{np+1}}{np+1}\right|_a^b\right]^{1/p}\left[\left.\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\left[\frac{(x-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}\right]\right|_a^b+\left.\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\left[(b-a)^{1/s}x-\frac{(x-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}\right]\right|_a^b\right]^{1/q} \\
=\ & \frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\frac{(b-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\left[(b-a)^{1/s}b-\frac{(b-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}-(b-a)^{1/s}a\right]\right]^{1/q}
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{1}{n!}(b-a)^{1/p}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}\left[(b-a)\frac{\left|f^{(n)}(b)\right|^q}{\frac{1+s}{s}}+\left|f^{(n)}(a)\right|^q\left[b-\frac{(b-a)}{\frac{1+s}{s}}-a\right]\right]^{1/q} \\
=\ & \frac{1}{n!}(b-a)^{1/p}L_{np}^n(a,b)\left[\frac{s(b-a)\left|f^{(n)}(b)\right|^q}{1+s}+\frac{s(b-a)\left|f^{(n)}(a)\right|^q}{1+s}\right]^{1/q} \\
=\ & \frac{1}{n!}(b-a)^{1/p}L_{np}^n(a,b)(b-a)^{1/q}\left(\frac{s}{1+s}\right)^{1/q}\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)^{1/q} \\
=\ & \frac{1}{n!}(b-a)L_{np}^n(a,b)\left[\frac{s\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)}{1+s}\right]^{1/q}.
\end{aligned}$$

□

Corollary 2.1. *Under the same assumptions given in Theorem 1,*
i) If we choose $s = 1$, then the inequality (2.1) becomes the following inequality,

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \leq \frac{b-a}{n!}L_{np}^n(a,b)A^{1/q}\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right).$$

ii) If we choose $s=1$, $n=1$, then the inequality (2.1) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq L_p(a,b)A^{1/q}(\left|f'(a)\right|^q,\left|f'(b)\right|^q).$$

iii) If we choose $s=1$, $n=1$, $q=1$,then the inequality (2.1) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq L_p(a,b)A(\left|f'(a)\right|,\left|f'(b)\right|).$$

Theorem 2. For $n\in\mathbb{N}$, let $f:I\subset[0,\infty)\longrightarrow\mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b\in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q\in L[a,b]$ for some fixed $s\in[0,1)$, then the following inequality holds;

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\\ \leq\ &\frac{b-a}{n!}L_n^{n\frac{q-1}{q}}(a,b)\left[\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q},\quad(2.2)\end{aligned}$$

where, $F\left(\frac{1}{s},n,x\right)=\int_a^b(x-a)^{1/s}x^n dx$.

Proof. From Lemma 1.2, using the well known Power mean integral inequality and the s-convexity in the third sense of $\left|f^{(n)}\right|^q$, we get

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\\ \leq\ &\frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\int_a^b x^n\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q+\left[\left(1-\left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right]^{1/q}\\ =\ &\frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\int_a^b x^n(x-a)^{1/s}dx+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b\left[(b-a)^{1/s}-(x-a)^{1/s}\right]x^n dx\right]^{1/q}\\ =\ &\frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)\int_a^b(x-a)^{1/s}x^n dx+\left|f^{(n)}(a)\right|^q\int_a^b x^n dx\right]^{1/q}\\ =\ &\frac{1}{n!}\left[\frac{x^{n+1}}{n+1}\Big|_a^b\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q\frac{x^{n+1}}{n+1}\Big|_a^b\right]^{1/q}\\ =\ &\frac{1}{n!}\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q\frac{b^{n+1}-a^{n+1}}{n+1}\right]^{1/q}\\ =\ &\frac{1}{n!}(b-a)^{1-1/q}L_n^{n\frac{q-1}{q}}(a,b)(b-a)^{1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s+1}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}\\ =\ &\frac{b-a}{n!}L_n^{n\frac{q-1}{q}}(a,b)\left[\left(\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s+1}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}.\end{aligned}$$

□

Corollary 2.2. *Under the same assumptions given in Theorem 2,*

i) If we choose $s=1$, then the inequality (2.2) becomes the following inequality,

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{n!}L_n^{n\frac{q-1}{q}}(a,b)\left[\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^2}(L_{n+1}^{n+1}(a,b)-aL_n^n(a,b))+\left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}.$$

ii) If we choose $s=1, n=1$, then the inequality (2.2) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\left(\frac{1}{b}\right)^{1/q}A^{1-1/q}(a,b)\left[(2b+a)\left|f'(b)\right|^q+(2a+b)\left|f'(a)\right|^q\right]^{1/q}.$$

iii) If we choose $s=1, n=1$ ve $q=1$, then the inequality (2.2) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(2b+a)\left|f'(b)\right|+(2a+b)\left|f'(a)\right|.$$

Theorem 3. For $n\in\mathbb{N}$, let $f: I\subset[0,\infty)\longrightarrow\mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b\in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q\in L[a,b]$ for some fixed $s\in[0,1)$, then the following inequality holds;

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{n!}\left[\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}G\left(\frac{1}{s},n,q,x\right)+\left|f^{(n)}(a)\right|^q L_{nq}^n(a,b)\right]^{1/q}, \tag{2.3}$$

where $G\left(\frac{1}{s},n,q,x\right)=\int_a^b x^{nq}(x-a)^{1/s}dx$.

Proof. From Lemma 1.2, using the well known Hölder integral inequality and the s-convexity in the third sense of $\left|f^{(n)}\right|^q$, we find,

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\\
\leq\ &\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx\right|\\
\leq\ &\frac{1}{n!}\left(\int_a^b 1^p dx\right)^{1/p}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{1/q}\\
\leq\ &\frac{1}{n!}\left(\int_a^b 1dx\right)^{1/p}\left(\int_a^b x^{nq}\left[\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q+\left(1-\left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right)^{1/q}\\
=\ &\frac{1}{n!}\left(\int_a^b dx\right)^{1/p}\left(\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b x^{nq}(x-a)^{1/s}dx+\left|f^{(n)}(a)\right|^q\int_a^b x^{nq}dx\right)^{1/q}\\
=\ &\frac{1}{n!}(b-a)^{1/p}\left((b-a)\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}G\left(\frac{1}{s},n,q,x\right)+\left|f^{(n)}(a)\right|^q(b-a)\left(\frac{b^{nq+1}-a^{nq+1}}{(b-a)(nq+1)}\right)\right)^{1/q}\\
=\ &\frac{1}{n!}(b-a)^{1/p}(b-a)^{1/q}\left(\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}G\left(\frac{1}{s},n,q,x\right)+\left|f^{(n)}(a)\right|^q L_{nq}^n(a,b)\right)^{1/q}.
\end{aligned}$$

□

Corollary 2.3. *Under the same assumptions given in Theorem 3,*
i) If we choose $s = 1$, then the inequality (2.3) becomes the following inequality,

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\\ \leq\ &\left[\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{n!}\left[L_{nq+1}^{nq+1}(a,b)-aL_{nq}^{nq}(a,b)\right]+\left|f^n(a)\right|^q L_{nq}^{nq}(a,b)\right]^{1/q}.\end{aligned}$$

ii) If we choose $s = 1, n = 1$, then the inequality (2.3) becomes the following inequality,

$$\begin{aligned}&\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\left[\left(\left|f'(b)\right|^q-\left|f'(a)\right|^q\right)\left[L_{q+1}^{q+1}(a,b)-aL_q^q(a,b)\right]+\left|f'(a)\right|^q L_q^q(a,b)\right]^{1/q}.\end{aligned}$$

iii) If we choose $s = 1, n = 1$ ve $q = 1$, then the inequality (2.3) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{b}\left[(2b+a)\left|f'(b)\right|+(2a+b)\left|f'(a)\right|\right].$$

Theorem 4. For $n \in \mathbb{N}$, let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be $n-$times differentiable function on I^o and let $a, b \in I^o$ with $a < b$. If $\left|f^{(n)}\right|^q$ for $q > 1$ is s-convex in the third sense function on $[a, b]$ and $\left|f^{(n)}\right|^q \in L[a, b]$ for some fixed $s \in [0, 1)$, then the following inequality holds;

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\leq\frac{(b-a)^{1/p}}{n!}2^{\frac{1-s^2}{qs^2}}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|. \tag{2.4}$$

Proof. From Lemma 1.2, the s-concavity in the third sense of $\left|f^{(n)}\right|^q$ and using the Hermite-Hadamard inequality for s-convex functions in the third sense [10] , we get

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\\ \leq\ &\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx\\ \leq\ &\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{1/p}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{1/q}\\ \leq\ &\frac{1}{n!}\left[\frac{x^{np+1}}{np+1}\Big|_a^b\right]^{1/p}\left[2^{\frac{1}{s^2}-1}\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|^q\right]^{1/q}\\ =\ &\frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}2^{\frac{1}{qs^2}-\frac{1}{q}}\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|\\ =\ &\frac{(b-a)^{1/p}}{n!}2^{\frac{1-s^2}{qs^2}}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|.\end{aligned}$$

□

Corollary 2.4. *Under the same assumptions given in Theorem 4,*

i) If we choose $s = 1$, then the inequality (2.4) becomes the following inequality,

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \leq \frac{(b-a)^{1/p}}{n!}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

ii) If we choose $s = 1, n = 1$, then the inequality (2.4) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{(b-a)^{1/p}}-\frac{1}{(b-a)^{1/p}}\int_a^b f(x)dx\right| \leq L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

3 CONCLUSION

The paper deals with some new integral inequalities that differ from the present results for n-times differantiable s-convex functions in the thirth sense. Similar studies have been done for different classes of abstract convex functions. By proving a new integral identity, researchers interested in convexity can derive new inequalities for this new class.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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