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Conformable fractional heat equation with fractional translation symmetry in both time and space

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Abstract

In this paper, we investigated the fractional heat equation with fractional translation in both time and position with different fractional orders. As examples, we considered a rod and an α-disk with an initial constant temperature and discussed their cooling processes in the examined formalism.

Keywords: Fractional derivative; heat equation; conformable symmetry.

1 Introduction

The theoretical investigation of fractional equations has received much attention in recent years [1]. For linear systems, equations with fractional derivatives serve as a crucial tool in the description and analysis of kinetic and transfer processes. For linear systems, fractional differentiation/integration operators with non-integer exponents can spontaneously overcome the irreversibility problem, and hence they can be employed to describe a wide range of kinetic/transfer phenomena. Therefore, we observe a growing interest in applying fractional calculus in diverse scientific fields of natural and applied sciences [1, 2, 3, 4, 5, 6, 7, 8]. Historically, Khalil, Al Horani, Yousef, and Sababheh were the first who introduce conformable fractional derivatives (CFD) [9]. Some applications of the CFD formalism can be found in the literature [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In a recent work, authors adopted this formalism to the quantum harmonic oscillator problem [27].

The CFD can be used for time, t, since it is always positive. Moreover, it is useful in position, x, because position space is defined in the interval of $(-\infty, \infty)$. Recently, Chung

and Hassanabadi [28] showed that the CFD is related to the fractional translation symmetry where they introduced the α-addition. They used the α-translation to explain the anomalous diffusion model where fractional order was shown to be related to the fractional power in the anomalous diffusion.

Some studies concerning the conformable fractional heat equation are given in Refs. [13, 25, 26]. For example in Ref. [13] the conformable fractional heat equation with CFD in time is examined. In Ref. [25] conformable fractional heat equation with CFD in time and space with same fractional order is investigated. More precisely, there they used the CFD in x and restrict the domain of x as $x > 0$. In another work, in Ref. [26], two dimensional conformable fractional heat equation with CFD in time only is dealt with.

In this paper, we discuss the fractional heat equation with fractional translation in both time and position with different fractional orders. Our model is well defined in $-\infty < x < \infty$ and unifies the results of Refs. [13, 25, 26]. Besides, we show that the conformable fractional heat equation possess the fractional translation symmetry. The paper is constructed as follows: In section 2, we briefly introduce the CFD with fractional translational invariance. Then, in section 3, we present the derivation of the conformable fractional heat equation. Next, in section 4, we derive non-dimensional conformable fractional heat equation. For applications, we consider a rod and an α-disk at a constant temperature in sections 5 and 6 and investigate their cooling processes by the introduced formalism, respectively. Finally, in the conclusion section, we summarize our findings.

2 CFD with fractional translational invariance

Let us consider the ordinary heat equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1}$$

where u and κ denote the temperature and thermal conductivity, respectively. The eq.(1) is invariant under the time translation, $t \to t + \Delta t$, and the space translation, $x \to x + \Delta x$. Thus, if $u(x,t)$ is a solution of the eq.(1), then $u(x+a, t+b)$ will also be a solution for constant a and b.

What will happen if translation invariance is broken and is replaced with a deformed translation? This was first studied in Ref. [28] where the authors found the fractional translation symmetry in the CFD. Here, we will adopt the different fractional order for CFD with respect to time t and CFD with respect to position x.

2.1 CFD with respect to time

Following Ref. [28] we introduce the fractional time derivative as

$$D_t^\beta F(t) = \lim_{t' \to t} \frac{F(t') - F(t)}{\frac{1}{\beta}(t'^\beta - t^\beta)} = t^{1-\beta} \partial_t F(t). \tag{2}$$

Let us introduce the β-addition and β-subtraction for $a \geq 0, b \geq 0$ as given in Ref. [28]

$$a \oplus_\beta b = (a^\beta + b^\beta)^{1/\beta}, \tag{3}$$

$$a \ominus_\beta b = \begin{cases} (a^\beta - b^\beta)^{1/\beta} & (a \geq b), \\ -(b^\beta - a^\beta)^{1/\beta} & (a < b). \end{cases} \tag{4}$$

Then, we rewrite the eq.(2) as

$$D_t^\beta F(t) = \lim_{t' \to t} \frac{F(t') - F(t)}{\frac{1}{\beta}(t' \ominus_\beta t)^\beta}. \tag{5}$$

Let us introduce the β-deformed time interval

$$\Delta_\beta t = t' \ominus_\beta t. \tag{6}$$

Then, we have

$$D_t^\beta F(t) = \lim_{\Delta_\beta t \to 0} \frac{F(t \oplus_\beta \Delta_\beta t) - F(t)}{\frac{1}{\beta}(\Delta_\beta t)^\beta}. \tag{7}$$

Here, we know that the fractional time derivative is invariant under the fractional β-translation in time,

$$D^\beta_{t \oplus_\beta \tau} = D_t^\beta. \tag{8}$$

2.2 CFD with respect to position

Unlike time, position can take a negative value. Thus, the fractional position derivative can be written as

$$D_x^\alpha F(x) = \lim_{x' \to x} \frac{F(x') - F(x)}{\frac{1}{\alpha}(|x'|^{\alpha-1}x' - |x|^{\alpha-1}x)} = |x|^{1-\alpha}\partial_x F(x). \tag{9}$$

By considering the α-addition and α-subtraction for a, b as given in Ref. [28]

$$a \oplus_\alpha b = |a|a|^{\alpha-1} + b|b|^{\alpha-1}|^{1/\alpha-1}(a|a|^{\alpha-1} + b|b|^{\alpha-1}), \tag{10}$$

$$a \ominus_\alpha b = |a|a|^{\alpha-1} - b|b|^{\alpha-1}|^{1/\alpha-1}(a|a|^{\alpha-1} - b|b|^{\alpha-1}), \tag{11}$$

we rewrite the eq.(9) in the following form

$$D_x^\alpha F(x) = \lim_{x' \to x} \frac{F(x') - F(x)}{\frac{1}{\alpha}|x' \ominus_\alpha x|^{\alpha-1}(x' \ominus_\alpha x)}. \tag{12}$$

Then, we introduce the α-deformed space interval

$$\Delta_\alpha x = x' \ominus_\beta x, \tag{13}$$

and we get

$$D_x^\alpha F(x) = \lim_{\Delta_\alpha x \to 0} \frac{F(x \oplus_\alpha \Delta_\alpha x) - F(x)}{\frac{1}{\alpha}|\Delta_\alpha x|^{\alpha-1}\Delta_\alpha x}. \tag{14}$$

Here, we know that the fractional position derivative is invariant under the fractional α-translation in position,

$$D^{\alpha}_{x\oplus_{\alpha}a} = D^{\alpha}_{x}. \tag{15}$$

The conformable fractional integral is then given by

$$\int d_{\alpha}xF(x) = \int dx\alpha|x|^{\alpha-1}F(x). \tag{16}$$

3 Derivation of conformable fractional heat equation

First let us derive the conformable fractional heat equation [29, 30] in one dimension (1D). To this end, we consider a rod of length L, and we assume that the heat flux is given by

$$J(x,t) = -\kappa D^{\alpha}_{x}u(x,t), \qquad 0 \leq x \leq L \tag{17}$$

We adopt the conformable fractional continuity equation so that it possesses the fractional translation symmetry,

$$D^{\beta}_{t}u(x,t) = -D^{\alpha}_{x}J(x,t), \tag{18}$$

which gives the conformable fractional heat equation in 1D in the form

$$D^{\beta}_{t}u(x,t) = \kappa(D^{\alpha}_{x})^{2}u(x,t). \tag{19}$$

In three dimension, conformable fractional heat equation reads

$$D^{\beta}_{t}u(x,t) = \kappa\left((D^{\alpha}_{x})^{2} + (D^{\alpha}_{y})^{2} + (D^{\alpha}_{z})^{2}\right)u(x,t), \tag{20}$$

where

$$D^{\beta}_{t} = t^{1-\beta}\partial_{t}, \tag{21}$$

$$D^{\alpha}_{x} = |x|^{1-\alpha}\partial_{x}, \tag{22}$$

$$D^{\alpha}_{y} = |y|^{1-\alpha}\partial_{y}, \tag{23}$$

$$D^{\alpha}_{z} = |z|^{1-\alpha}\partial_{z}. \tag{24}$$

It is worth noting that, the case of $\alpha = 1$ in 1D is discussed in Ref. [13], while the case of $\alpha = \beta$ in 1D with $x > 0$ is examined in Ref. [25], and the case of $\alpha = 1$ in two dimensions (2D) is explored in Ref. [26].

4 Non-dimensional conformable fractional heat equation

As it is well-known, in a metal bar with an inhomogeneous temperature, thermal energy, in other words heat, flows from regions of higher temperature to regions of lower temperature.

Here, we assume that the heat transfer is governed by the conformable fractional heat equation instead of the traditional one.

Dimensional (or physical) terms in the eq.(19) or the eq.(20) can be changed into the non-dimensional ones by introducing the appropriate scale such as characteristic length, time and temperature. Then, the conformable fractional heat equation in 1D reads

$$D_t^{\beta} u(x,t) = (D_x^{\alpha})^2 u(x,t). \qquad 0 \leq x \leq 1 \tag{25}$$

Next, we consider the following boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = u_0(x), \tag{26}$$

and seek a solution to the conformable fractional heat eq.(25) of the form

$$u(x,t) = X(x)T(t). \tag{27}$$

Inserting the eq.(27) into the eq.(25) yields

$$\frac{(D_x^{\alpha})^2 X}{X} = \frac{D_t^{\beta} T}{T}. \tag{28}$$

We solve the x part with $X(0) = X(1) = 0$, we find

$$X_n(x) = \sin\left(n\pi x^{\alpha}\right), \qquad n = 1, 2, 3, \cdots \tag{29}$$

Similarly, we solve the t part, we obtain

$$T_n(t) = e^{-\left(\alpha^2 n^2 \pi^2 t\right)^{\beta}}. \tag{30}$$

Therefore, the solution of the eq.(25) is

$$u_n(x,t) = B_n e^{-\left(\alpha^2 n^2 \pi^2 t\right)^{\beta}} \sin\left(n\pi x^{\alpha}\right) \tag{31}$$

Applying the principle of superposition, the general solution reads

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\alpha^2 n^2 \pi^2 t\right)^{\beta}} \sin\left(n\pi x^{\alpha}\right). \tag{32}$$

Considering the initial condition $u(x,0) = u_0(x)$, we have

$$u_0(x) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi x^{\alpha}\right). \tag{33}$$

From the orthogonality relation of the trigonometric functions, we have

$$B_n = 2\alpha \int_0^1 \sin\left(n\pi x^{\alpha}\right) u_0(x) d_{\alpha} x, \tag{34}$$

where we used

$$\int_0^1 \sin\left(n\pi x^{\alpha}\right) \sin\left(m\pi x^{\alpha}\right) d_{\alpha} x = \frac{1}{2\alpha} \delta_{nm}. \tag{35}$$

$$\int_0^1 \cos\left(n\pi x^{\alpha}\right) \cos\left(m\pi x^{\alpha}\right) d_{\alpha} x = \frac{1}{2\alpha} \delta_{nm}. \tag{36}$$

$$\int_0^1 \cos\left(n\pi x^{\alpha}\right) \sin\left(m\pi x^{\alpha}\right) d_{\alpha} x = 0. \tag{37}$$

5 Rod's cooling

We consider a metal rod with an initial constant temperature distribution $u_0(x)$, i.e.

$$u_0(x) = u_0 = \text{constant}, \tag{38}$$

In this case we have

$$B_n = 2u_0 \left(\frac{1-(-1)^n}{n\pi} \right). \tag{39}$$

Thus, the solution reads

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-\left(\alpha^2(2n-1)^2\pi^2 t\right)^\beta} \sin\left((2n-1)\pi x^\alpha\right), \tag{40}$$

or

$$u(x,t) = \frac{4u_0}{\pi} \left[e^{-\left(\alpha^2\pi^2 t\right)^\beta} \sin\left(\pi x^\alpha\right) + \frac{1}{3} e^{-\left(9\alpha^2\pi^2 t\right)^\beta} \sin\left(3\pi x^\alpha\right) + \cdots \right]. \tag{41}$$

The ratio of the first and second terms in eq.(41) is

$$\frac{|\text{second term}|}{|\text{first term}|} = \frac{1}{3} e^{-(9^\beta - 1)(\alpha^2\pi^2 t)^\beta} \frac{|\sin 3\pi x^\alpha|}{|\sin \pi x^\alpha|} \leq e^{-(9^\beta-1)(\alpha^2\pi^2 t)^\beta}, \tag{42}$$

where we used

$$|\sin nt| \leq n|\sin t|. \tag{43}$$

For $t \geq \frac{1}{\alpha^2\pi^2}$, the ratio gives

$$\frac{|\text{second term}|}{|\text{first term}|} \leq e^{-(9^\beta-1)}. \tag{44}$$

In Fig.1, we demonstrate the plot of $f(\beta) = e^{-(9^\beta-1)}$ versus β for $0.9 \leq \beta \leq 1.1$. We notice that for small $|\beta - 1|$, the second term is negligible, hence we find

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-\left(\alpha^2\pi^2 t\right)^\beta} \sin\left(\pi x^\alpha\right). \tag{45}$$

5.1 Spatial temperature profiles

Now let us consider fixed time. Here we consider the time $t = 2/(\alpha\pi)^2$. Then, we have

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-\left(2\alpha^2\right)^\beta} \sin\left(\pi x^\alpha\right). \tag{46}$$

This function takes its maximum value at $x = (1/2)^\alpha$. Thus, the center of a rod is not a line of symmetry unless $\alpha = 1$. Fig.2 shows the plot of u versus x with $u_0 = \pi/4$ for $\alpha = \beta = 1$ (Red), $\alpha = 0.9, \beta = 0.9$ (Green), $\alpha = 0.9, \beta = 1.1$ (Brown), $\alpha = 1.1, \beta = 0.9$ (Pink) and $\alpha = 1.1, \beta = 1.1$ (Purple).

5.2 Profiles of temperature over time

In order to understand how the heat dissipation changes with time in the same location, we take $x = x_0$ in the equation 45. We find

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-\left(\alpha^2\pi^2 t\right)^\beta} \sin\left(\pi x_0^\alpha\right). \tag{47}$$

In Fig.3, we display the plot of u versus t with $u_0 = \pi/4, x_0 = (1/2)^\alpha$ for $\alpha = \beta = 1$ (Red), $\alpha = 0.9, \beta = 0.9$ (Green), $\alpha = 0.9, \beta = 1.1$ (Brown), $\alpha = 1.1, \beta = 0.9$ (Pink) and $\alpha = 1.1, \beta = 1.1$ (Purple).

6 α-disk's cooling

Here, we consider a 2D problem, namely the cooling of an α-disk from a constant initial temperature. Therefore, we employ the following 2D conformable fractional heat equation:

$$D_t^\beta u(x,t) = \kappa \left((D_x^\alpha)^2 + (D_y^\alpha)^2\right) u(x,t). \tag{48}$$

It is worth noting that the fractional Laplacian is not invariant under the ordinary $SO(2)$ group. Instead, it is invariant under the fractional version of $SO(2)$ group which induces the invariant quantity

$$r_\alpha = (|x|^{2\alpha} + |y|^{2\alpha})^{\frac{1}{2\alpha}}. \tag{49}$$

Therefore, we consider the α-disk with α-radius, R_α,

$$|x|^{2\alpha} + |y|^{2\alpha} = R_\alpha^{2\alpha}. \tag{50}$$

In Fig.4 we present the plot of α-disk with unit α-radius for $\alpha = 1$ (Red), $\alpha = 0.9$ (Green) and $\alpha = 1.1$ (Brown). Then, we consider the non-dimensional case, and write

$$D_t^\beta u(x,t) = \left((D_x^\alpha)^2 + (D_y^\alpha)^2\right) u(x,t), \quad 0 \le r_\alpha \le 1, \tag{51}$$

where α-polar coordinate reads

$$\frac{1}{\alpha}|x|^{\alpha-1}x = \frac{1}{\alpha} r_\alpha^\alpha \cos\theta, \tag{52}$$

$$\frac{1}{\alpha}|y|^{\alpha-1}y = \frac{1}{\alpha} r_\alpha^\alpha \sin\theta. \tag{53}$$

Then, the eq.(51) reads

$$D_t^\beta u(r_\alpha,t) = \left((D_{r_\alpha}^\alpha)^2 + \frac{1}{r_\alpha^\alpha} D_{r_\alpha}^\alpha\right) u(r_\alpha,t). \tag{54}$$

Now, let us impose the following initial conditions:

$$u(1,t) = 0, \quad u(r_\alpha,0) = u_0 = \text{constant}. \tag{55}$$

Then, the solution reads

$$u(r_\alpha, t) = \sum_{n=1}^{\infty} C_n J_0(\gamma_n r_\alpha^\alpha) e^{-\gamma_n^2 \left(\frac{t^\beta}{\beta}\right)}, \tag{56}$$

where γ_n is the n-th zero of $J_0(x)$. From the second relation of the eq.(54), we have

$$u_0 = \sum_{n=1}^{\infty} C_n J_0(\gamma_n r_\alpha^\alpha), \tag{57}$$

which gives

$$C_n = \frac{2u_0}{\gamma_n J_1(\gamma_n)}. \tag{58}$$

Therefore, the solution reads

$$u(r_\alpha, t) = 2u_0 \sum_{n=1}^{\infty} \frac{1}{\gamma_n J_1(\gamma_n)} J_0(\gamma_n r_\alpha^\alpha) e^{-\gamma_n^2 \left(\frac{t^\beta}{\beta}\right)}. \tag{59}$$

This result can be approximated to

$$u(r_\alpha, t) \approx 2u_0 \frac{1}{\gamma_1 J_1(\gamma_1)} J_0(\gamma_1 r_\alpha^\alpha) e^{-\gamma_1^2 \left(\frac{t^\beta}{\beta}\right)}, \tag{60}$$

where $\gamma_1 \approx 2.40483$.

Now let us consider a fixed time value, $t = \left(\frac{2\beta}{\gamma_1^2}\right)^{1/\beta}$. Then, we have

$$u(r_\alpha, t) \approx 2u_0 e^{-2} \frac{1}{\gamma_1 J_1(\gamma_1)} J_0(\gamma_1 r_\alpha^\alpha) \tag{61}$$

In Fig.5 we depict the plot of u versus r_α for $\alpha = 1$ (Red), $\alpha = 0.9$ (Green) and $\alpha = 1.1$ (Brown). When $\alpha < 1$ (or $\alpha > 1$), the temperature decreases faster (or slower) than $\alpha = 1$ from the center to the boundary.

7 Conclusion

In this paper, we introduce the conformable fraction derivatives for time and space which might have different fractional orders. Using these derivatives, we obtain conformable fractional heat equation which is well defined in $-\infty < x < \infty$. We observe that the latter heat equation is the unified result of Refs.[13, 25]. Besides, we show that the conformable fractional heat equation possesses the fractional translation symmetry in both time and space. Then, as an application we consider a 1D metal rod with a constant initial temperature and study its cooling process. For the fixed time, the temperature is shown to have a maxima at $x = (1/2)^\alpha$. Thus, we conclude that the center of a rod is not a line of symmetry unless $\alpha = 1$. Finally, we consider a 2D problem and examine the cooling process of a α-disk which has a constant initial temperature. We find that for $\alpha < 1$ (or $\alpha > 1$) the temperature decreases faster (or slower) than $\alpha = 1$ from the center to the boundary.

Declarations

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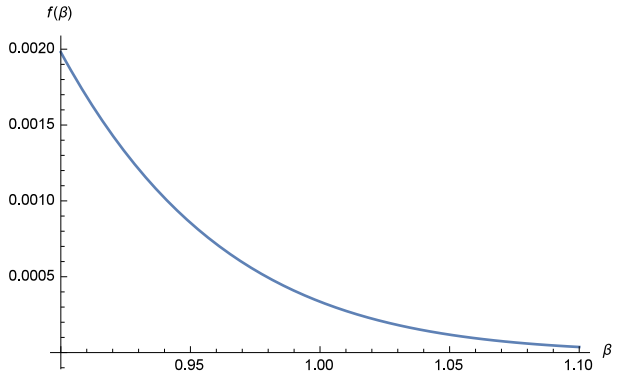


Figure 1: Plot of $f(\beta) = e^{-(9^{\beta}-1)}$ versus β for $0.9 \leq \beta \leq 1.1$.

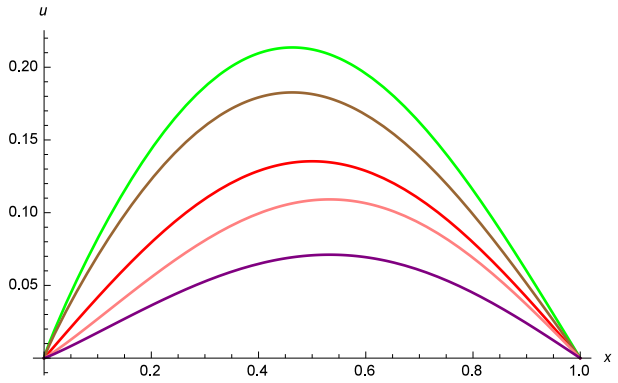


Figure 2: Plot of u versus x with $u_0 = \pi/4$ for $\alpha = \beta = 1$ (Red), $\alpha = 0.9, \beta = 0.9$ (Green), $\alpha = 0.9, \beta = 1.1$ (Brown), $\alpha = 1.1, \beta = 0.9$ (Pink) and $\alpha = 1.1, \beta = 1.1$ (Purple).

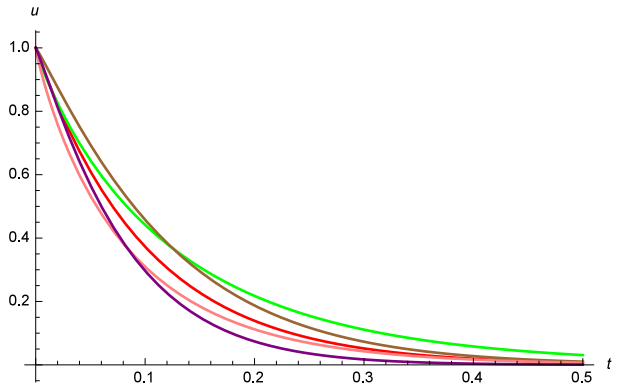


Figure 3: Plot of u versus t with $u_0 = \pi/4, x_0 = (1/2)^{\alpha}$ for $\alpha = \beta = 1$ (Red), $\alpha = 0.9, \beta = 0.9$ (Green), $\alpha = 0.9, \beta = 1.1$ (Brown), $\alpha = 1.1, \beta = 0.9$ (Pink) and $\alpha = 1.1, \beta = 1.1$ (Purple).

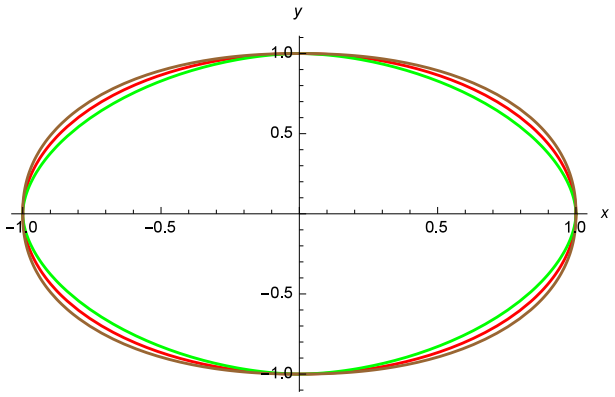


Figure 4: Plot of α-disk with unit α-radius for $\alpha = 1$ (Red), $\alpha = 0.9$ (Green) and $\alpha = 1.1$ (Brown).

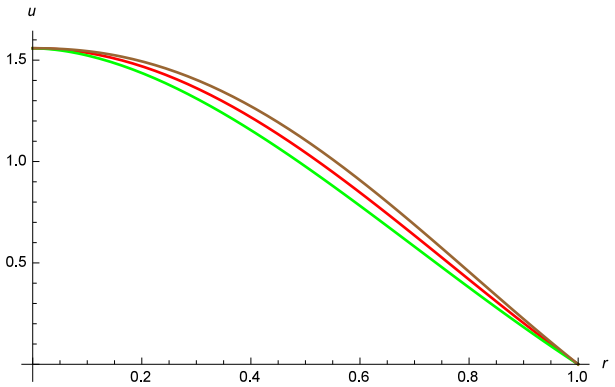


Figure 5: Plot of u versus r_α for $\alpha = 1$ (Red), $\alpha = 0.9$ (Green) and $\alpha = 1.1$ (Brown).