HYBRID NUMBERS

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Abstract. In this study, we define a new non-commutative number system called hybrid numbers. This number system can be accepted as a generalization of the complex $(\mathbf{i}^2 = -1)$, hyperbolic $(\mathbf{h}^2 = 1)$ and dual number $(\varepsilon^2 = 0)$ systems. A hybrid number is a number created with any combination of the complex, hyperbolic and dual numbers satisfying the relation $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$. Because these numbers are a composition of dual, complex and hyperbolic numbers, we think that it would be better to call them hybrid numbers instead of the generalized complex numbers. In this paper, we give some algebraic and geometric properties of this number set with some classifications. In addition, we examined the roots of a hybrid number according to its type and character.

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1. Introduction

Complex, hyperbolic and dual numbers are well known two dimensional number systems. Especially in the last century, a lot of researchers deal with the geometric and physical applications of these numbers. Just as the geometry of the Euclidean plane can be described with complex numbers, the geometry of the Minkowski plane and Galilean plane can be described with hyperbolic numbers

$$\mathbb{P} = \{\mathbf{z} = x + \mathbf{h}y : \mathbf{h}^2 = 1,\ x, y \in \mathbb{R}\},$$

and dual numbers

$$\mathbb{D} = \{\mathbf{z} = x + \varepsilon y : \ \varepsilon^2 = 0,\ x, y \in \mathbb{R}\},$$

respectively [58]. Hyperbolic numbers can be named as "spacetime", "double", "perplex" and "split-complex numbers" in literature ([3], [14], [45], [46]). The complex, dual and hyperbolic numbers are simply referred to as the hypercomplex numbers [5]. It is well known that the group of Euclidean rotations SO(2) is isomorphic to the group U(1) of unit complex numbers

$$e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta.$$

Namely, the geometrical meaning of multiplying by $e^{i\theta}$ means a rotation of the plane. Some geometric applications of complex numbers can be find in [1] and [59]. Similarly, the group of Lorentzian rotations SO(1,1) is isomorphic to the group of unit spacelike hyperbolic numbers

$$e^{\mathbf{h}\theta} = \cosh\theta + \mathbf{h}\sinh\theta.$$

This rotation can be viewed as hyperbolic rotation. Thus, multiplying by $e^{\mathbf{h}\theta}$ means a map of hyperbolic numbers into itself which preserves the Lorentzian metric [7], [8], [9], [47], [48]. On the other hand, the Galilean rotations can be interpreted with dual numbers. The concept of a rotation in the dual number plane is equivalent to a vertical shear mapping since

$$(1 + x\varepsilon)(1 + y\varepsilon) = 1 + (x + y)\varepsilon.$$

Using the equality $\varepsilon^n = 0$ for any integer $n > 1$, we can write the Euler formula

$$e^{\varepsilon\theta} = 1 + \varepsilon\theta$$

for dual numbers. So, multiplying by $e^{\varepsilon\theta}$ is a map of dual numbers into itself which preserves the Galilean metric. This rotation can be named as parabolic rotation [27], [29]. Comparing complex, dual and hyperbolic numbers in terms of algebraic properties and geometric interpretations like planar transformations can be found in the Rooney's paper [47].

Dual numbers were introduced in 1873 by William Clifford. In 1903, E. Study defined dual numbers as dual angles to specify the relation between two lines in the Euclidean space and proved that there exists a one-to-one correspondence between the points of the dual unit sphere and the directed lines of Euclidean 3-space [53]. Dual numbers are extensively used in the quantum mechanics ([19], [21], [20], [24], [25]) and classic mechanics of screws [11].

Matrix representations of complex, hyperbolic and dual numbers become

$$a + \mathbf{i}b \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a + \mathbf{h}b \leftrightarrow \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ and } a + \varepsilon b \leftrightarrow \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

respectively ([27], [29]). So, the complex, hyperbolic and dual units are

$$\mathbf{i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{h} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \varepsilon \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.1}$$

Here, each mapping is an isomorphism because the operations of addition and multiplication correspond to the matrix addition and multiplication. Thus, we have the matrix forms of Euler formulas for complex, hyperbolic and dual numbers as explained below :

$$\cos\theta + \mathbf{i}\sin\theta = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad (1.2)$$

$$\cosh\theta + \mathbf{h}\sinh\theta = \exp\begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \quad (1.3)$$

$$1 + \varepsilon\theta = \exp\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}.$$

These matrices are the rotation matrices in the Euclidean, the Lorentzian and the Galilean plane respectively.

In abstract algebra, the complex, the hyperbolic and the dual numbers can be described as the quotient of the polynomial ring $\mathbb{R}[x]$ by the ideal generated by the polynomials x^2+1, x^2-1 and x^2 respectively. That is,

$$\mathbb{C} = \mathbb{R}[x] / \langle x^2+1 \rangle, \ \mathbb{P} = \mathbb{R}[x] / \langle x^2-1 \rangle, \ \mathbb{D} = \mathbb{R}[x] / \langle x^2 \rangle.$$

Although the set of complex numbers is a field, it is not true for the hyperbolic and dual numbers. They are commutative rings. These three sets of numbers can be summarized briefly by means of the following table.

	Complex Numbers	Hyperbolic Numbers	Dual Numbers
Property	$\mathbf{z}=a+\mathbf{i}b$, $\mathbf{i}^2=-1$	$\mathbf{z}=a+\mathbf{h}b$, $\mathbf{h}^2=1$	$\mathbf{z}=a+\varepsilon b$, $\varepsilon^2=0$
Conjugate	$\overline{\mathbf{z}} = a - \mathbf{i}b$	$\overline{\mathbf{z}} = a - \mathbf{h}b$	$\overline{\mathbf{z}} = a - \varepsilon b$
Norm	$\lvert\mathbf{z}\rvert=\sqrt{a^2+b^2}$	$\lvert\mathbf{z}\rvert=\sqrt{a^2-b^2}$	$\lvert\mathbf{z}\rvert=\lvert a\rvert$
Geometry	Euclidean Geo.	Lorentzian Geo.	Galilean Geo.
Circle	$x^2+y^2=r^2$	$x^2-y^2=\pm r^2$	$\lvert x\rvert=r$
Rotation Type	Elliptic	Hyperbolic	Parabolic
Euler Formula	$e^{\mathbf{i}\theta}$=cosθ+isinθ	$e^{\mathbf{h}\theta}$=coshθ+hsinhθ	$e^{\varepsilon\theta} = 1+\varepsilon\theta$
Argument	arg$\mathbf{z}$=arctan$\frac{b}{a}$	arg$\mathbf{z}$=ln$\frac{\lvert a+b\rvert}{\sqrt{\lvert a^2-b^2\rvert}}$	arg$\mathbf{z} = \frac{b}{a}$

For a detailed information about dual and hyperbolic numbers we can refer to the excellent books of Yaglom [58], Vladimir V. Kisil [26], Catoni and others [8], and Fisher [17]. In addition, the algebraic properties and the geometric applications of these numbers can be found in the Erlangen Program research papers of Vladimir V. Kisil ([27], [28], [29], [30]). Geometry of Mobius transformations for hypercomplex numbers can be found in Kisil's book. Besides, Brewer deals with a new cross-ratio of hypercomplex numbers based on projective geometry [5]. Hyperbolic numbers are applicable to nonlinear differential equations ([42], [43], [44]). Some geometrical, physical and algebraic properties and applications of hypercomplex numbers can be found in several papers : [2], [3], [4], [6], [12], [17], [32], [33], [52], [54], [55], [56]. Whereas the concept of complex structure is used in the Euclidean differential geometry, the concept of hyperbolic structure obtained with hyperbolic numbers can be used in the Lorentzian differential geometry, as an instance see [50] and [51].

Complex, hyperbolic and dual numbers can be defined as Clifford algebras using elliptic, hyperbolic and parabolic bilinear forms respectively.

A Clifford algebra as a four dimensional space which is spanned by $1, e_0, e_1, e_0e_1$ with non-commutative multiplication defined by the following identities :

$$e_0^2 = -1,\ e_1^2 = \sigma = \begin{cases} -1, & \text{for } C\ell\,(e) \text{ - Elliptic case} \\ 0, & \text{for } C\ell\,(p) \text{ - Parabolic case} \\ 1, & \text{for } C\ell\,(h) \text{ - Hyperbolic case} \end{cases},\ e_0e_1 + e_1e_0 = 0.$$

It can be used the notation $C\ell\,(\sigma)$ for any of these three algebras, assuming the values for $\sigma = -1,\ 0,\ 1$. Even subalgebra of $C\ell^+\,(\sigma)$ is isomorphic to complex numbers, hyperbolic numbers and dual numbers for σ is -1, 1 and 0 respectively. For example, the two dimensional even subalgebra $C\ell^+\,(-1)$ spanned by $\{1, e_0e_1\}$ is isomorphic to complex numbers and we have

$$\mathbf{i}^2 = (e_0e_1)^2 = (e_0e_1)(e_0e_1) = e_0\,(e_1e_0)\,e_1 = -e_0^2e_1^2 = -1.$$

Similarly, it can be seen the isomorphisms $C\ell^+\,(0) \cong \mathbb{D}$ and $C\ell^+\,(1) \cong \mathbb{P}$ [30].

Generalizations of Complex Numbers

With the use of a large area of complex numbers, it is desirable to generalize complex numbers. The most famous generalization of complex numbers is quaternions. In 1843, William Rowan Hamilton described the set of quaternions

$$\mathbb{H} = \left\{\mathbf{z} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\right\}$$

And James Cockle defined coquaternions (split quaternions)

$$\overline{\mathbb{H}} = \left\{\mathbf{z} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : \mathbf{i}^2 = -1,\ \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1\right\}$$

in 1849 [10]. Quaternions and coquaternions are used to define 3D Euclidean and Lorentzian rotations respectively. A similar relation can be presented between split quaternions and relations in the Minkowski 3-space, like the one between the quaternions and the rotations in the Euclidean 3-space. [37]. A natural generalization of quaternions and coquaternions with their geometrical applications can be found in the papers [39] and [49]. In these papers, elliptical and hyperbolic quaternions were defined.

In 1963, Yaglom described some special quaternions that he called degenerate quaternions, degenerate pseudoquaternions, and double degenerate quaternions. Multiplication tables of these special quaternions as follows [59].

	$\mathbf{i}$	ε_1	ε_2
$\mathbf{i}$	-1	ε_2	$-\varepsilon_1$
ε_1	$-\varepsilon_2$	0	0
ε_2	ε_1	0	0

Degenerate Quaternions

	$\mathbf{h}$	ε_1	ε_2
$\mathbf{h}$	1	ε_2	ε_1
ε_1	$-\varepsilon_2$	0	0
ε_2	$-\varepsilon_1$	0	0

Degenerate Pseudoquaternions

	ε_1	ε_2	ε_3
ε_1	0	ε_3	0
ε_2	$-\varepsilon_3$	0	0
ε_3	0	0	0

Doubly Degenerate Pseudoquaternions

In 1998, Fjelstad and Gal generalized the 2-dimensional number systems to higher dimensions using a very natural way. For example, n-dimensional hyperbolic number system constructed a basis $1, h_1, h_2, ..., h_{n-1}$ where, $h_i^2 = 1$, $h_i \notin \mathbb{R}$ for all i. They defined n-dimensional dual and hyperbolic numbers with some algebraic properties ([15], [16]).

In 2000, Olariu defined a different generalization of n-dimensional complex numbers naming them twocomplex numbers, threecomplex numbers. Olariu used the name twocomplex numbers for hyperbolic numbers. He studied the geometrical and the algebraic properties of these numbers. The set of threecomplex numbers was defined as

$$\mathbb{C}_3 = \left\{ \mathbf{z} = a + \mathbf{h}b + \mathbf{k}c : a, b, c \in \mathbb{R} \text{ and } \mathbf{h}^2 = \mathbf{k},\ \mathbf{k}^2 = \mathbf{h},\ \mathbf{hk} = 1 \right\}.$$

In 2004, Anthony Harkin and Joseph Harkin generalized two dimensional complex numbers as

$$\mathbb{C}p = \{\mathbf{z} = x + \mathbf{i}y : \mathbf{i}^2 = p,\ x, y \in \mathbb{R}\}.$$

They gave some trigonometric relations for this generalization [23]. After that Catoni and others defined two dimensional hypercomplex numbers as

$$\mathbb{C}_{\alpha,\beta} = \{\mathbf{z} = x + \mathbf{i}y : \mathbf{i}^2 = \alpha + \mathbf{i}\beta,\ x, y, \alpha, \beta \in \mathbb{R},\ \mathbf{i} \notin \mathbb{R}\},$$

and extended the relationship between these numbers and Euclidean and Semi-Euclidean geometry [7]. This generalization is also expressible as a quotient ring $\mathbb{R}[x] / \left\langle x^2 - \beta x - \alpha \right\rangle$.

In 2017, Zaripov presented a theory of commutative two-dimensional conformal hyperbolic numbers as a generalization of the theory of hyperbolic numbers [60].

In this study, we define a new generalization of complex, hyperbolic and dual numbers different from above generalizations. These generalizations can not be studied until now. In this generalization, we present a system of such numbers that consists of all three number systems together. Although this new number system appears to be four-dimensional, it can be viewed as a two-dimensional set of numbers, since it can be represented in a generalized two-dimensional plane, which we call it hybridian plane in $\mathbb{R}^4$. Because the new set of numbers contains the three number systems and the units $\mathbf{i}, \mathbf{h}$ and ε are mentioned above, we thought that it would be better to name them the hybrid numbers instead of the generalized numbers. Since the matrix representation of dual unit given above is not consistent with the hybrid numbers, we will use a different matrix representation for it. In particular, since we are used to having the matrix representations of complex and hyperbolic numbers which are frequently used in the literature, they have not been changed. Because, these preferred matrix representations correspond to well known rotations matrices in the Euclidean and Lorentzian plane (1.3), (1.2). Since the traditional matrix representations (1.1) of $\mathbf{i}, \varepsilon$ and $\mathbf{h}$ with the unit matrix are not a base for the set of 2×2 matrices, we use the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

as the matrix representation of ε. This matrix representation has not been chosen at random. It is taken to be appropriate to the isomorphism given in Theorem 3.1. If a different isomorphisms are taken, the matrix representations of $\mathbf{i}, \varepsilon$ and $\mathbf{h}$ will change. Accordingly, the multiplication and definition of the hybrid numbers defined in the second section will also change. In this study, a matrix isomorphism has been chosen to be the most suitable one for the literature and in particular to overlap with the matrix representations of complex and hyperbolic numbers. Thus, a hybrid number system based on the following matrix representations has been established. These four matrices are a base of a 2×2 matrix set.

$$\mathbf{1} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \varepsilon \leftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \mathbf{h} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This paper is organized in the following way : In section 2, we define the hybrid numbers and explained some basic properties of this number system. We classify these numbers with two different aspects {complike=elliptic, hiperlike=hiperbolic, duallike=parabolic} and {spacelike, timelike, lightlike}. Here, the words complike, hiperlike and duallike means that the numbers similar to complex, hyperbolic and dual numbers respectively. In fact, these classifications are related to the matrix representation of the hybrid numbers given in the third section. It is possible to classify hybrid numbers according to the determinant and the discriminant of the characteristic equation of this matrix. Since we give the matrix representation of hybrid numbers in the next sections, we will express the classifications of hybrid numbers according to the conjugate and the norm as in the hyperbolic numbers and split quaternions ([3], [37]). In Section 3, we deal with some representations of the hybrid numbers. The matrix, polar and exponential representations of the hybrid numbers are examined separately according to their types and characters. In the next sections, de Moivre formula for the hybrid numbers is proved and the methods for finding the roots of a hybrid number are studied separately according to its character and type. At last, we give an elementary linear algebraic application for a hybrid number.

2. Hybrid Numbers

The set of hybrid numbers will be denoted by $\mathbb{K}$ and contains complex, dual and hyperbolic numbers as well as combined and mixed states of these types of three numbers. The corresponding geometry of hybrid numbers will be most general geometry including their combination as well as the Euclidean, Minkowski, and Galilean plane geometries. We will call the geometry corresponding to the hybrid numbers as the Hybridian plane geometry. This plane is a two dimensional subspace of $\mathbb{R}^4$. We will classify the Hybridian geometry as elliptical, hyperbolic, or parabolic, and we will study it separately according to the type of the hybrid number. Also, as in complex numbers, we will prove the De Moivre's formula for the hybrid numbers according to the their classifications. It is known that a commutative two dimensional unital algebra generated by a 2×2 matrix is isomorphic to either complex, dual or hyperbolic numbers [31]. Due to the set of hybrid numbers is a two-dimensional commutative algebra spanned by 1 and $b\mathbf{i} + c\varepsilon + d\mathbf{h}$, it is isomorphic to one of the complex, dual or hyperbolic numbers.

Definition 2.1. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K}=\left\{a+b\mathbf{i}+c\varepsilon+d\mathbf{h}: a,b,c,d\in\mathbb{R},\ \mathbf{i}^2=-1,\ \varepsilon^2=0,\ \mathbf{h}^2=1,\ \mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}\right\}.$$

This set of numbers can be thought as a set of quadruplet. And the real, complex, dual and hyperbolic units are defined as

$$1\leftrightarrow(1,0,0,0),\ \mathbf{i}\leftrightarrow(0,1,0,0),\ \varepsilon\leftrightarrow(0,0,1,0),\ \mathbf{h}\leftrightarrow(0,0,0,1)$$

respectively. We will call these units as hybrid units. For the hybrid number $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$, the real number a is called the scalar part and is denoted by $S(\mathbf{Z})$. The part $b\mathbf{i}+c\varepsilon+d\mathbf{h}$ is also called the vector part and is denoted by $V(\mathbf{Z})$.

Remark 2.2. In the definition of hybrid number, it is conceivable how the relation $\mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}$ between complex, hyperbolic and dual units are found or why these equations are preferred. We see very clearly the consistency of these equalities with the aid of the isomorphism established between the hybrid numbers and the set of 2 by 2 matrices. The basic relation $\mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}$ is defined to be appropriate to the isomorphism given in Theorem 3.1. That is, it is possible to define the matrix isomorphism first, and then to define hybrid numbers as appropriate for this isomorphism. However, as already mentioned, a matrix isomorphism and the set of hybrid numbers has been established in this study to be most appropriate for the literature and in particular to overlap with the matrix representations of complex and hyperbolic numbers.

Operations in the Hybrid Numbers

Two hybrid numbers are equal if all their components are equal, one by one. The sum of two hybrid numbers is defined by summing their components. Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{Z}$ is $-\mathbf{Z}$, which is defined as having all the components of $\mathbf{Z}$ changed in their signs. This implies that, $(\mathbb{K},+)$ is an Abelian group.

The Hybridian product

$$\mathbf{ZW}=(a_1+b_1\mathbf{i}+c_1\varepsilon+d_1\mathbf{h})(a_2+b_2\mathbf{i}+c_2\varepsilon+d_2\mathbf{h})$$

is obtained by distributing the terms on the right as in ordinary algebra, preserving that the multiplication order of the units and then writing the values of followings replacing each product of units by the equalities $\mathbf{i}^2=-1$, $\varepsilon^2=0$, $\mathbf{h}^2=1$, $\mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}$. Using these equalities we can find the product of any two hybrid units. For example, let's find $\varepsilon\mathbf{i}$. For this, let's multiply $\mathbf{hi}+\varepsilon+\mathbf{i}=0$ by $\mathbf{i}$ from the right. Thus, we get $\varepsilon\mathbf{i}=\mathbf{h}+1$. If we continue in a similar way, we get the following multiplication table.

$\cdot$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
$\mathbf{1}$	1	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$\mathbf{1}-\boldsymbol{h}$	$\boldsymbol{\varepsilon}+\boldsymbol{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}+\mathbf{1}$	0	$-\boldsymbol{\varepsilon}$
$\boldsymbol{h}$	$\boldsymbol{h}$	$-\boldsymbol{\varepsilon}-\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	1

(2.1)

Accordingly, we will use this table for the multiplication of any two hybrid numbers. This table shows us that the multiplication operation in the hybrid numbers is not commutative. But it has the property of associativity. It is not easy to remember this table. Also, the lack of commutativity makes it difficult to do multiplication. We will make this multiplication in much easier way with the aid of an isomorphism, which we will later identify between hybrid numbers and 2×2 matrices.

Norm and Classification of the Hybrid Numbers

Definition 2.3. The conjugate of a hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, denoted by $\overline{\mathbf{Z}}$, is defined as

$$\overline{\mathbf{Z}} = S(\mathbf{Z}) - V(\mathbf{Z}) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$$

as in the quaternions. The conjugate of the sum of hybrid numbers is equal to the sum of their conjugates. Also, according to the hybridian product, we have $\mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z}$. The real number

$$\mathcal{C}(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2$$

is called the character of the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$.

Definition 2.4. For the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, the vector

$$\mathcal{V}_{\mathbf{Z}} = (a, (b-c), c, d)$$

is called **the vector representation of Z**. The second component of this vector will be better understood in the following parts. Also, we can write as

$$\mathcal{C}(\mathbf{Z}) = a^2 + (b-c)^2 - c^2 - d^2 = -\langle \mathcal{V}_{\mathbf{Z}}, \mathcal{V}_{\mathbf{Z}} \rangle_{\mathbb{E}_2^4}$$

with the signature $(-,-,+,+)$ in $\mathbb{E}_2^4$. We say that a hybrid number $\mathbf{Z}$ is spacelike, timelike or lightlike if $\mathcal{C}(\mathbf{Z}) < 0$, $\mathcal{C}(\mathbf{Z}) > 0$ or $\mathcal{C}(\mathbf{Z}) = 0$ respectively. These are called the characters of the hybrid number $\mathbf{Z}$. The real number $\sqrt{|\mathcal{C}(\mathbf{Z})|}$ will be called **the norm of the hybrid number Z** and will be denoted by $\|\mathbf{Z}\|$.

Remark 2.5. Let $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. According to definition of the norm of a hybrid number we have

$$\|\mathbf{Z}\| = \sqrt{\left|\mathbf{Z}\overline{\mathbf{Z}}\right|} = \sqrt{|\mathcal{C}(\mathbf{Z})|} = \sqrt{\left|a^2 + (b-c)^2 - c^2 - d^2\right|}.$$

This definition of the norm is a generalized norm that overlaps with the definitions of the norms in the complex, hyperbolic and dual numbers. Indeed,

1. If $\mathbf{Z}$ is a complex number (c=d=0), then $\|\mathbf{Z}\| = \sqrt{\left|\mathbf{Z}\overline{\mathbf{Z}}\right|} = \sqrt{a^2 + b^2}$,

2. If $\mathbf{Z}$ is a hyperbolic number (b=c=0), then $\|\mathbf{Z}\| = \sqrt{\left|\mathbf{Z}\overline{\mathbf{Z}}\right|} = \sqrt{|a^2 - d^2|}$,

3. If $\mathbf{Z}$ is a dual number (b=d=0), then $\|\mathbf{Z}\| = \sqrt{a^2} = |a|$.

Definition 2.6. For the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, the vector,

$$\mathcal{E}_{\mathbf{Z}} = ((b-c), c, d)$$

is called the **hybrid vector** of the number of $\mathbf{Z}$ and this vector will also be taken as a vector of Minkowski 3-space $\mathbb{E}_1^3$. It can be written as

$$\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = -(b-c)^2 + c^2 + d^2 = \langle \mathcal{E}_{\mathbf{Z}}, \mathcal{E}_{\mathbf{Z}} \rangle_{\mathbb{E}_1^3}.$$

Thus, we say that a hybrid number $\mathbf{Z}$ is elliptic (complike), hyperbolic (hiperlike) or parabolic (duallike), if $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) < 0$, $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) > 0$ or $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = 0$ respectively. These are called **the types of the hybrid number Z**. The real number $\sqrt{|\mathcal{C}_{\mathcal{E}}(\mathbf{Z})|}$ will be called as the norm of the hybrid vector of $\mathbf{Z}$ and will be denoted by $\mathcal{N}(\mathbf{Z})$.

As in the split quaternions, the hybrid vector of a hybrid number can be timelike, spacelike, or null. Also, the hybrid vector of a spacelike hybrid number will certainly be spacelike. We can actually see that

$$\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = \langle \mathcal{E}_{\mathbf{Z}}, \mathcal{E}_{\mathbf{Z}} \rangle_{\mathbb{E}_1^3} = -(b-c)^2 + c^2 + d^2 > 0$$

from the inequality

$$\mathcal{C}(\mathbf{Z}) < 0 \Rightarrow a^2 + (b-c)^2 - c^2 - d^2 < 0 \Rightarrow -(b-c)^2 + c^2 + d^2 > a^2 > 0.$$

It means that the type of a spacelike hybrid number is definitely hyperbolic. Similarly, the hybrid vector of a lightlike hybrid number is strictly a spacelike vector if the scalar part of it is nonzero, and a null vector if the scalar part is zero. That is, the type of a lightlike hybrid number is either hyperbolic or parabolic. Thus, we can give the following table. According to this table, if a hybrid number is elliptic, the its character is definitely timelike.

Spacelike	**Lightlike**	**Timelike**
Hyperbolic (Hiperlike)	Hyperbolic	Hyperbolic
	Parabolic (Duallike)	Parabolic
		Elliptic (Complike)

In the other side, using the hybridian product of hybrid numbers, one can show that the equality

$$\mathcal{C}(\mathbf{Z}_1\mathbf{Z}_2) = \mathcal{C}(\mathbf{Z}_1)\,\mathcal{C}(\mathbf{Z}_2).$$

So, the timelike hybrid numbers form a group according to the multiplication operation an it is denoted by

$$\mathbb{TK} = \{\mathbf{Z} \in \mathbb{K} : \mathcal{C}(\mathbf{Z}) > 0\}.$$

According to the hybridian product we have the following table.

$\cdot$	**Spacelike**	**Timelike**	**Lightlike**
Spacelike	Timelike	Spacelike	Lightlike
Timelike	Spacelike	Timelike	Lightlike
Lightlike	Lightlike	Lightlike	Lightlike

Any hybrid number can be shown in a two dimensional coordinate system like the Cartesian coordinate system. For this purpose, a coordinate plane constructed with the real axis and the hybrid vector. We choose the real part of the hybrid number as the real axis and the value of $\sqrt{|\mathcal{C}_{\mathcal{E}}(\mathbf{Z})|} = \mathcal{N}(\mathbf{Z})$ as the hybrid axis. That is, the value of scalar part of a hybrid number $\mathbf{Z}$ is called the x-coordinate or abscissa and the value of $\sqrt{|\mathcal{C}_{\mathcal{E}}(\mathbf{Z})|}$ is called the y-coordinate or ordinate. The coordinate system created in this way is called the hybridian coordinate system. Accordingly, the hybrid coordinates of a hybrid number $\mathbf{Z} = \mathbf{Z} = \mathbf{a} + \mathbf{bi} + \mathbf{c}\varepsilon + \mathbf{dh} = x + y\mathbf{V}$ are denoted by

$$\mathbf{Z} = (x, y) = (a, \mathcal{N}(\mathbf{Z})) = (a, \sqrt{\left|-(b-c)^2 + c^2 + d^2\right|})$$

where $\mathbf{V}$ is the hybridian vector of $\mathbf{Z}$. The hybridian plane can be accepted as a subspace of the four dimensional semi-Euclidean space.

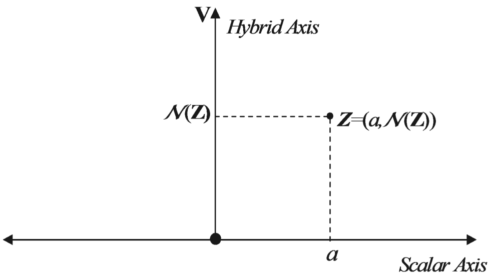

FIGURE 1. Hybridian Coordinates

For example, the hybrid coordinates of the number $\mathbf{Z} = 1 + 3\mathbf{i} + 1\varepsilon + 1\mathbf{h}$ is

$$\mathbf{Z} = \left(1, \sqrt{2}\right)$$

where the hybrid axis is formed by the unit hybrid vector $\mathbf{V} = \dfrac{3\mathbf{i} + 1\varepsilon + 1\mathbf{h}}{\sqrt{2}}$.

Definition 2.7. The inverse of the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, $\|\mathbf{Z}\| \neq 0$ is defined as

$$\mathbf{Z}^{-1} = \frac{\overline{\mathbf{Z}}}{\mathcal{C}(\mathbf{Z})}.$$

Accordingly, lightlike hybrid numbers have no inverse.

For example, the inverse of the hybrid number $\mathbf{Z} = 2 + 5\mathbf{i} + 3\varepsilon + \mathbf{h}$ is

$$\mathbf{Z}^{-1} = \frac{2 - 5\mathbf{i} - 3\varepsilon - \mathbf{h}}{-2}.$$

Thus, we can write the following theorem, from the above properties.

Theorem 2.8. *The set of hybrid numbers is a non-commutative ring with respect to the addition and multiplication operations.*

Scalar and Vector Product for the Hybrid Numbers

The scalar product in the hybrid numbers is defined as follows.

$$\begin{aligned} g(p,q) &: \mathbb{K}\times\mathbb{K}\to\mathbb{R} \\ g(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{\mathbf{Z}_1\overline{\mathbf{Z}}_2+\mathbf{Z}_2\overline{\mathbf{Z}}_1}{2}=a_1a_2+b_1b_2-b_1c_2-b_2c_1-d_1d_2 \end{aligned}$$

where $\mathbf{Z}_1=a_1+b_1\mathbf{i}+c_1\varepsilon+d_1\mathbf{h}$ and $\mathbf{Z}_2=a_2+b_2\mathbf{i}+c_2\varepsilon+d_2\mathbf{h}$. This scalar product is a nondegenerate, symmetric bilinear form. The scalar product of the hybrid numbers is a generalized scalar product for the complex, hyperbolic and dual number systems. However, with this new scalar product, we can now also refer to the scalar product between a complex number and a hyperbolic number. According to this, we can give the table

$g(\mathbf{Z}_1,\mathbf{Z}_2)$	Complex $\mathbf{Z}_2$	Hyperbolic $\mathbf{Z}_2$	Dual $\mathbf{Z}_2$
Complex $\mathbf{Z}_1$	$a_1a_2+b_1b_2$	a_1a_2	$a_1a_2-b_1c_2$
Hyperbolic $\mathbf{Z}_1$	a_1a_2	$a_1a_2-d_1d_2$	a_1a_2
Dual $\mathbf{Z}_1$	$a_1a_2-b_2c_1$	a_1a_2	a_1a_2

where $\mathbf{Z}_1=a_1+b_1\mathbf{i}+c_1\varepsilon+d_1\mathbf{h}$ and $\mathbf{Z}_2=a_2+b_2\mathbf{i}+c_2\varepsilon+d_2\mathbf{h}$. For example, the scalar product of the complex number $\mathbf{Z}_1=3+2\mathbf{i}$ and the dual number $\mathbf{Z}_2=4-5\varepsilon$ is 22. We can write the following equalities.

$$\begin{gathered} g(\mathbf{1},\mathbf{1})=g(\mathbf{i},\mathbf{i})=1,\ g(\mathbf{h},\mathbf{h})=-1 \text{ and } g(\varepsilon,\varepsilon)=0 \\ g(\mathbf{1},\mathbf{i})=g(\mathbf{1},\varepsilon)=g(\mathbf{1},\mathbf{h})=g(\mathbf{i},\mathbf{h})=g(\varepsilon,\mathbf{h})=0 \text{ and } g(\mathbf{i},\varepsilon)=-1. \end{gathered}$$

On the other hand, the vector product of two hybrid numbers $\mathbf{Z}_1$ and $\mathbf{Z}_2$ defined as

$$\begin{aligned} \times &: \mathbb{K}\times\mathbb{K}\to\mathbb{K} \\ \mathbf{Z}_1\times\mathbf{Z}_2 &= \frac{\mathbf{Z}_1\overline{\mathbf{Z}}_2-\mathbf{Z}_2\overline{\mathbf{Z}}_1}{2}. \end{aligned}$$

According to this definition, the Hermitian product of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is given with the equation

$$\mathbf{Z}_1\overline{\mathbf{Z}}_2=g(\mathbf{Z}_1,\mathbf{Z}_2)+\mathbf{Z}_1\times\mathbf{Z}_2.$$

Thus, we can satisfy the vector product table for hybrid numbers as

$\times$	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{0}$	$-\mathbf{i}$	$-\varepsilon$	$-\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$\mathbf{0}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$
ε	ε	$-\mathbf{h}$	$\mathbf{0}$	ε
$\mathbf{h}$	$\mathbf{h}$	$\varepsilon+\mathbf{i}$	$-\varepsilon$	$\mathbf{0}$

.

3. Representations of Hybrid Numbers

In this Section, we will express the matrix, polar and the exponential representations of the hybrid numbers according to their characters and types. These representations are important especially for proving the de Moivre formula and for finding the roots of hybrid numbers which are given in the next sections.

3.1. The Matrix Representation of Hybrid Numbers

A matrix representation for a hybrid number is especially important in order to facilitate multiplication of hybrid numbers. By defining an isomorphism between 2×2 matrices and hybrid numbers, we can easily multiply the hybrid numbers and prove many of their features. On the other hand, hybrid numbers can also be defined by considering the matrix representation. We can classify the hybrid numbers, with respect to determinant and discriminant of the characteristic equation of the 2×2 corresponding matrix.

Theorem 3.1. *The hybrid number ring* $\mathbb{K}$ *is isomorphic to the ring of real* 2×2 *matrices* $\mathbb{M}_{2\times 2}$.

Proof. Let's define the map $\varphi : \mathbb{K} \to \mathbb{M}_{2\times 2}$ where

$$\varphi\left(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}\right) = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}$$

for $\mathbf{Z} = a+b\mathbf{i}+c\varepsilon+d\mathbf{h} \in \mathbb{K}$. We will show that this map is a ring isomorphism. Taking into account the addition and multiplication operation table defined for the hybrid numbers before, it can be easily seen that the equalities

$$\begin{aligned} \varphi\left(\mathbf{Z}_1\mathbf{Z}_2\right) &= \varphi\left(\mathbf{Z}_1\right)\varphi\left(\mathbf{Z}_2\right) \\ \varphi\left(\mathbf{Z}_1+\mathbf{Z}_2\right) &= \varphi\left(\mathbf{Z}_1\right)+\varphi\left(\mathbf{Z}_2\right) \end{aligned}$$

are satisfied. Now, let's see that φ is bijective. If

$$\varphi\left(\mathbf{Z}_1\right) = \varphi\left(\mathbf{Z}_2\right)$$

for $\mathbf{Z}_1 = a_1+b_1\mathbf{i}+c_1\varepsilon+d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2+b_2\mathbf{i}+c_2\varepsilon+d_2\mathbf{h}$, then the equality of the two matrices results in that $c_1 = c_2$, $a_1 = a_2$, $b_1 = b_2$, $d_1 = d_2$. That is $\mathbf{Z}_1 = \mathbf{Z}_2$ and φ is injective. On the other hand, for any 2 by 2 real matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

there is a hybrid number

$$\mathbf{Z} = \left(\frac{a+d}{2}\right) + \left(\frac{a+b-c-d}{2}\right)\mathbf{i} + \left(\frac{a-d}{2}\right)\varepsilon + \left(\frac{b+c}{2}\right)\mathbf{h} \tag{3.1}$$

where $\varphi\left(\mathbf{Z}\right) = A$. Thus, φ is a ring isomorphism. □

Definition 3.2. The matrix $\varphi\left(\mathbf{Z}\right) \in \mathbb{M}_{2\times 2}\left(\mathbb{R}\right)$ is called the hybrid matrix corresponding to the hybrid number $\mathbf{Z}$.

According to the above isomorphism, we have the matrix representations

$$\varphi(1)=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \varphi(\mathbf{i})=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\ \varphi(\varepsilon)=\begin{bmatrix}1&-1\\1&-1\end{bmatrix},\ \varphi(\mathbf{h})=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

These four matrices form a basis for the vector space of 2×2 matrices. So, each 2 by 2 matrix can be written as a composite of unit, dual, complex, and hyperbolic numbers. Notice that we will use the matrix

$$\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$$

instead of $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ for the dual unit ε, on the contrary to literature.

With the aid of these four matrices, the multiplication of the hybrid numbers described above, can also be easily handled. For example, using the equalities

$$\begin{aligned}\varphi(\mathbf{ih}) &= \varphi(\mathbf{i})\,\varphi(\mathbf{h})\\ &= \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\\ &= \begin{bmatrix}1&0\\0&-1\end{bmatrix}\\ &= \varphi(\varepsilon)+\varphi(\mathbf{i})\\ &= \varphi(\varepsilon+\mathbf{i}),\end{aligned}$$

we have $\mathbf{ih}=\varepsilon+\mathbf{i}$. Thus, we can easily do operations and calculations in the hybrid numbers using the corresponding matrices

$$1\leftrightarrow\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \mathbf{i}\leftrightarrow\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \varepsilon\leftrightarrow\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\text{ and }\mathbf{h}\leftrightarrow\begin{bmatrix}0&1\\-1&0\end{bmatrix}.$$

Theorem 3.3. *Let A be a 2 by 2 real matrix corresponding to the hybrid number $\mathbf{Z}$, then there are the following equalities.*

$$\begin{aligned}\mathcal{C}(\mathbf{Z}) &= \det A \ \textit{and}\ \|\mathbf{Z}\|=\sqrt{|\det A|};\\ \mathcal{C}_{\mathcal{E}}(\mathbf{Z}) &= \frac{(trA)^2-4\det A}{4}.\end{aligned}$$

Proof. Let $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ be a hybrid number. Then, we have

$$\begin{aligned}|\det A| &= \begin{vmatrix}a+c & b-c+d\\ c-b+d & a-c\end{vmatrix}\\ &= \left|a^2+b^2-2cb-d^2\right|\\ &= \left|\mathbf{Z}\overline{\mathbf{Z}}\right|\\ &= \|\mathbf{Z}\|^2.\end{aligned}$$

The second equality is clear from $\mathcal{C}_{\mathcal{E}}(\mathbf{Z})=a^2-\left(a^2+(b-c)^2-c^2-d^2\right)$. □

Corollary 3.4. *The inverse of a hybrid number $\mathbf{Z}\in\mathbb{K}$ exists if and only if* $\det(\varphi(\mathbf{Z}))\neq 0$.

Corollary 3.5. $\triangle_A = (trA)^2 - 4\det A$ *is equal to discriminant of the characteristic polynomial of a* 2×2 *real matrix* A*. So,* $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = \triangle_A/4$.

Corollary 3.6. *The classification of hybrid numbers depends entirely on the determinant and the trace of the* 2×2 *corresponding matrix.*

Remark 3.7. The establishment of hyperbolic numbers is entirely based on the matrix isomorphism. If a different isomorphism were used, the basic equations between complex, hyperbolic and dual units would also change. For example, if we take the isomorphism

$$\varphi_1 : \mathbb{K}_1 \to \mathbb{M}_{2\times 2},\ \varphi_1(a,b,c,d) = \begin{pmatrix} a+d & b+c \\ -b & a-d \end{pmatrix},$$

the matrix representations of units will be

$$\mathbf{1} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varepsilon \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{h} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, the fundamental relations can be taken as $\mathbf{hi} = -\mathbf{ih} = \mathbf{2}\varepsilon - \mathbf{i}$. Therefore, a new set of hybrid numbers is defined as,

$$\mathbb{K}_1 = \left\{a+b\mathbf{i}+c\varepsilon+d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = 2\varepsilon\text{-}\mathbf{i}\right\}.$$

But in this case, since the representation of the hyperbolic unit is not compatible with the traditional representation (1.1) in the literature, we would encounter the equality

$$\cosh\theta + \mathbf{h}\sinh\theta = \begin{pmatrix} \cosh\theta + \sinh\theta & 0 \\ 0 & \cosh\theta - \sinh\theta \end{pmatrix}.$$

This would also confront us with some inconsistencies and confusion, especially in subjects such as the rotation matrix on the Lorentzian plane and the geometric applications of hyperbolic numbers. For this reason, an isomorphism is used in this study in such a way that the matrix representation of hyperbolic and complex numbers and the rotation matrices of the Euclidean and Lorentzian planes do not change. So, it has been seen that the most convenient relations for defining the hybrid number are $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$. In the other side, if the relations $\mathbf{hi} = -\mathbf{ih} = \mathbf{2}\varepsilon - \mathbf{i}$ had been used, we would also encounter rational expressions such as $\mathbf{i}\varepsilon = (\mathbf{h} - 1)/2$ in the multiplication table. However, as you can see in the table (2.1), the table consists only of units.

3.2. The Polar Representations of Hybrid Numbers

Now, we give the polar representation of a hybrid number with respect to their type and character, separately. This representation will depend on whether the hybrid number is elliptic, hyperbolic, or parabolic. On the other hand, if the hybrid number is spacelike, timelike or lightlike, the polar representation will change. That is, the polar representation of a hybrid number depends on the type of the hybrid vector and the casual character. For this reason, we will give the polar representations of the hybrid numbers via three different theorems. These theorems also confirm that any two-dimensional unitary algebra is isomorphic to either dual, hyperbolic, or complex numbers [31].

Theorem 3.8. *Each elliptic hybrid number can be written as*

$$\mathbf{Z} = \|\mathbf{Z}\| \left(\cos\theta + \mathbf{V}_0 \sin\theta\right)$$

where $\mathbf{V}_0 = \dfrac{\mathcal{V}(\mathbf{Z})}{\mathcal{N}(\mathbf{Z})}$ *and* $\mathbf{V}_0^2 = -1$.

Proof. If the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$ is elliptic, the hybrid vector $\mathcal{E}_{\mathbf{Z}} = ((b-c), c, d) \in \mathbb{R}^3$ of $\mathbf{Z}$ is a timelike vector. Every elliptic hybrid number is timelike. That is, $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = -(b-c)^2 + c^2 + d^2 < 0$ and $\mathcal{C}(\mathbf{Z}) = a^2 + (b-c)^2 - c^2 - d^2 > 0$. Thus, we can write

$$\|\mathbf{Z}\| = \sqrt{a^2 + (b-c)^2 - c^2 - d^2} \text{ and } \mathcal{N}(\mathbf{Z}) = \sqrt{(b-c)^2 - c^2 - d^2}.$$

If we take

$$\cos\theta = \frac{a}{\|\mathbf{Z}\|},\ \ \sin\theta = \frac{\mathcal{N}(\mathbf{Z})}{\|\mathbf{Z}\|} \text{ and } \mathbf{V}_0 = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(\mathbf{Z})},$$

we get the polar form $\mathbf{Z} = \|\mathbf{Z}\| \left(\cos\theta + \mathbf{V}_0 \sin\theta\right)$. □

Theorem 3.9. *Each hyperbolic hybrid number* $\mathbf{Z}$ *can be written as*

$$\begin{array}{ll} \mathbf{Z} = \|\mathbf{Z}\| \left(\sinh\theta + \mathbf{V}_0 \cosh\theta\right), & \textit{If } \mathbf{Z} \textit{ is spacelike,} \\ \mathbf{Z} = \pm \|\mathbf{Z}\| \left(\cosh\theta + \mathbf{V}_0 \sinh\theta\right), & \textit{If } \mathbf{Z} \textit{ is timelike,} \\ \mathbf{Z} = a\left(1 + \mathbf{V}_0\right) & \textit{If } \mathbf{Z} \textit{ is lightlike,} \end{array}$$

where $\mathbf{V}_0 = \dfrac{\mathcal{V}(\mathbf{Z})}{\mathcal{N}(\mathbf{Z})}$ *and* $\mathbf{V}_0^2 = 1$.

Proof. If the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is hyperbolic, then the hybrid vector $\mathcal{E}_{\mathbf{Z}} = ((b-c), c, d) \in \mathbb{R}^3$ of $\mathbf{Z}$ is spacelike. It means that $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) > 0$ and $\mathcal{N}(\mathbf{Z}) = \sqrt{-(b-c)^2 + c^2 + d^2}$. On the other side, casual character of a hybrid number can be spacelike, timelike or lightlike. Therefore, we must examine it in three separate cases.

1) If $\mathbf{Z}$ is a spacelike hyperbolic hybrid number, then we have $C(\mathbf{Z}) < 0$ and $\|\mathbf{Z}\| = \sqrt{-a^2 - (b-c)^2 + c^2 + d^2}$. Therefore, $\mathbf{Z}$ can be written as

$$\mathbf{Z} = \|\mathbf{Z}\| \left(\sinh\theta + \mathbf{V}_0 \cosh\theta\right)$$

where

$$\sinh\theta = \frac{a}{\|\mathbf{Z}\|},\ \ \cosh\theta = \frac{\mathcal{N}(\mathbf{Z})}{\|\mathbf{Z}\|} \text{ and } \mathbf{V}_0 = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(\mathbf{Z})}.$$

2) If $\mathbf{Z}$ is a timelike hyperbolic hybrid number, then we have $C(\mathbf{Z}) > 0$ and $\|\mathbf{Z}\| = \sqrt{a^2 + (b-c)^2 - c^2 - d^2}$. In this case, $\mathbf{Z}$ can be written as

$$\mathbf{Z} = \|\mathbf{Z}\| \left(\cosh\theta + \mathbf{V}_0 \sinh\theta\right)$$

where

$$\cosh\theta = \frac{a}{\|\mathbf{Z}\|},\ \ \sinh\theta = \frac{\mathcal{N}(\mathbf{Z})}{\|\mathbf{Z}\|} \text{ and } \mathbf{V}_0 = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(\mathbf{Z})}.$$

3) If $\mathbf{Z}$ is a lightlike hyperbolic hybrid number, then we have $C(\mathbf{Z}) = 0$ and $\|\mathbf{Z}\| = 0$. Then, $\mathcal{N}(\mathbf{Z}) = a$ and $\mathbf{Z}$ can be written in the form

$$\mathbf{Z} = a(1 + \mathbf{V}_0)$$

where $\mathbf{V}_0 = \dfrac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(\mathbf{Z})} = \dfrac{\mathbf{V}(Z)}{a}$ and $\mathbf{V}_0^2 = 1$. $\square$

Example 1. Let's find the polar forms of the hybrid numbers
i) $\mathbf{Z}_1 = 3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h}$,
ii) $\mathbf{Z}_2 = 3 + 2\mathbf{i} + \varepsilon + 3\mathbf{h}$ and
iii) $\mathbf{Z}_3 = 1 + \mathbf{i} + 2\varepsilon + 3\mathbf{h}$.
i) $\mathbf{Z}_1$ is a timelike hyperbolic hybrid number since $\mathcal{C}(\mathbf{Z}_1) = 5 > 0$ and $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}_1) = 4 > 0$. So, we can write

$$\mathbf{Z}_1 = \sqrt{5}(\cosh\theta + \mathbf{V}_0 \sinh\theta)$$

where,

$$\cosh\theta = \frac{3}{\sqrt{5}},\ \sinh\theta = \frac{2}{\sqrt{5}} \text{ and } \mathbf{V}_0 = \frac{2\mathbf{i} + 1\varepsilon + 2\mathbf{h}}{2},\ \mathbf{V}_0^2 = 1.$$

ii) $\mathbf{Z}_2$ is a lightlike hyperbolic hybrid number since $\mathcal{C}(\mathbf{Z}_2) = 0$ and $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}_2) = 9 > 0$. So, we can write

$$\mathbf{Z}_2 = 3(1 + \mathbf{V}_0)$$

where $\mathbf{V}_0 = \dfrac{2\mathbf{i} + 1\varepsilon + 3\mathbf{h}}{3}$, $\mathbf{V}_0^2 = 1$.
iii) $\mathbf{Z}_3$ is a spacelike hyperbolic hybrid number since $\mathcal{C}(\mathbf{Z}_3) = -11 < 0$ and $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}_3) = 12 > 0$. So, we can write

$$\mathbf{Z}_3 = \sqrt{11}(\sinh\theta + \mathbf{V}_0 \cosh\theta)$$

where,

$$\cosh\theta = \frac{\sqrt{12}}{\sqrt{11}},\ \sinh\theta = \frac{1}{\sqrt{11}} \text{ and } \mathbf{V}_0 = \frac{2\mathbf{i} + 1\varepsilon + 2\mathbf{h}}{2},\ \mathbf{V}_0^2 = 1.$$

Polar form of the non-lightlike hyperbolic numbers can be expressed as the following corollary in only one form.

Corollary 3.10. *Each non-lightlike hyperbolic hybrid number* $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$ *can be written as*

$$\mathbf{Z} = k\rho(\cosh\theta + \mathbf{V}\sinh\theta)$$

where

$$k = \begin{cases} 1 & \mathbf{Z} \textit{ is timelike and } a > 0, \\ -1 & \mathbf{Z} \textit{ is timelike and } a < 0, \\ \mathbf{V} & \mathbf{Z} \textit{ is spacelike and } a > 0, \\ -\mathbf{V} & \mathbf{Z} \textit{ is spacelike and } a < 0, \end{cases}$$

and

$$\rho = \|\mathbf{Z}\|,\ \theta = \ln\left|\frac{a + \mathcal{N}(\mathbf{Z})}{\rho}\right| \textit{ and } \mathbf{V} = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(\mathbf{Z})}.$$

Theorem 3.11. *Each parabolic hybrid number can be written as*

$$\mathbf{Z} = \|\mathbf{Z}\| \left(\epsilon + \mathbf{V}_0\right)$$

where $\mathbf{V}_0 = \dfrac{\mathcal{V}(\mathbf{Z})}{\|\mathbf{Z}\|}$ *and* $\mathbf{V}_0^2 = 0$, $\epsilon = sgn\left(S\left(\mathbf{Z}\right)\right)$.

Proof. If the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is parabolic, then the hybrid vector $\mathcal{E}_{\mathbf{Z}} = ((b-c), c, d) \in \mathbb{R}^3$ of $\mathbf{Z}$ will be lightlike. It means that $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = 0$ and $\|\mathbf{Z}\| = |a|$. Thus, depending the sign of a, the hybrid number $\mathbf{Z}$ can be written as

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} = \|\mathbf{Z}\| \left(\epsilon + \mathbf{V}_0\right)$$

where $\mathbf{V}_0 = \dfrac{\mathcal{V}(\mathbf{Z})}{\|\mathbf{Z}\|}$ and $\mathbf{V}_0^2 = 0$. □

Example 2. $\mathbf{Z} = 2 + 8\mathbf{i} + 3\varepsilon + 4\mathbf{h}$ is a parabolic timelike hybrid number since $\mathcal{C}(\mathbf{Z}) = 4 > 0$ and $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = 0$. Due to $a = 2$, we can write

$$\mathbf{Z} = 2\left(1 + \mathbf{V}_0\right)$$

where $\mathbf{V}_0 = \dfrac{8\mathbf{i} + 3\varepsilon + 4\mathbf{h}}{2}$, $\mathbf{V}_0^2 = 0$.

Definition 3.12. The argument of a non-lightlike hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$ is defined as

$$\theta = \begin{cases} \pi - \arctan \dfrac{\mathcal{N}(\mathbf{Z})}{|a|} & a < 0 \\ \arctan \dfrac{\mathcal{N}(\mathbf{Z})}{|a|} & a > 0 \end{cases}$$

and

$$\theta = \ln \frac{|a + \mathcal{N}(\mathbf{Z})|}{\|\mathbf{Z}\|} \text{ and } \theta = \frac{c}{\|\mathbf{Z}\|}$$

for elliptic, hyperbolic and parabolic hybrid numbers respectively. This value is the angle between the position vector of the number $\mathbf{Z}$ and the hybrid axis or the real axis, depending on whether the character of $\mathbf{Z}$ is spacelike or timelike, respectively.

Example 3. Let's find the argument of the hyperbolic hybrid number $\mathbf{Z} = -3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h}$. We have $\|\mathbf{Z}\| = \sqrt{5}$, $\mathcal{N}(\mathbf{Z}) = 2$ and,

$$\theta = \ln \frac{|a + \mathcal{N}(\mathbf{Z})|}{\|\mathbf{Z}\|} = \ln \frac{\sqrt{5}}{5}.$$

The hybridian coordinates of $\mathbf{Z}$ is $(-3, 2)$, and this number lies on the hyperbola $x^2 - y^2 = 5$. The hyperbolic angle between the position vector of $\mathbf{Z}$ and

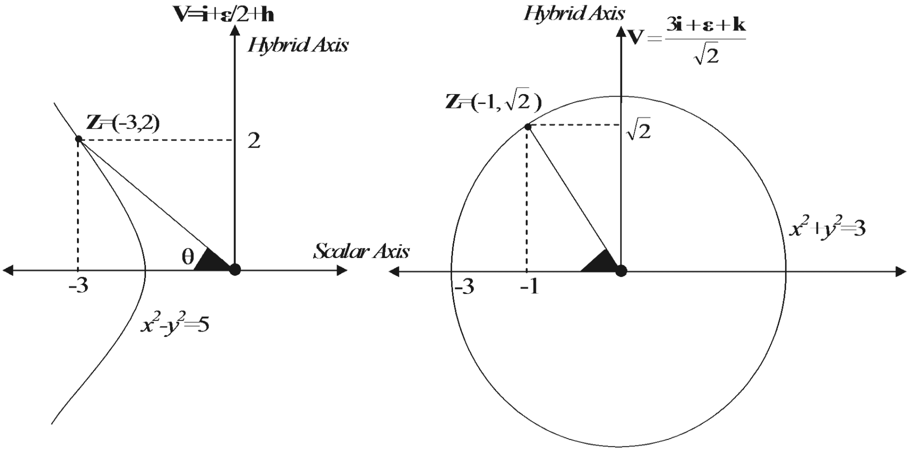

FIGURE 2. Hyperbolic and Elliptic Arguments

real axis is $\theta = \ln\sqrt{5}/5$ (Figure 2-a). Similarly, the argument of the timelike elliptic hybrid number $\mathbf{Z}{=}{-1}{+}3\mathbf{i} + 1\varepsilon{+}1\mathbf{h}$ is

$$\theta = \pi - \arctan\frac{\mathcal{N}\left(\mathbf{Z}\right)}{a} = \pi - \arctan\sqrt{2}.$$

The hybridian coordinates of $\mathbf{Z}$ is $\left(-1, \sqrt{2}\right)$ and this number lies on the circle $x^2 + y^2 = 3$. Also, this number is a point that makes the circular angle $\theta = \pi - \arctan\sqrt{2}$ with the real axis (Figure 2-b). Finally, let's give an example for argument of a parabolic hybrid number. In the example 2, we found that $\mathbf{Z} = 2 + 8\mathbf{i} + 3\varepsilon + 4\mathbf{h}$ $=2 + 3\mathbf{V}$ where $\mathbf{V}_0 = \dfrac{8\mathbf{i} + 3\varepsilon + 4\mathbf{h}}{3}$, $\mathbf{V}_0^2 = 0$. Thus, $\theta = \dfrac{3}{2}$ and $\mathbf{Z} = (2, 3)$ can be shown in the hybridian coordinate system as

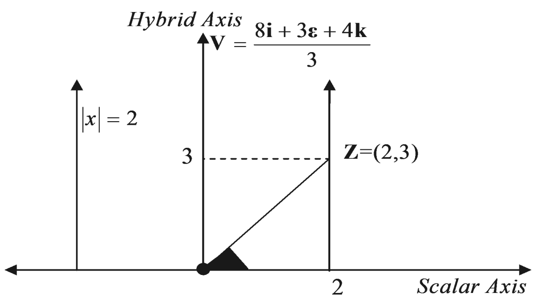

FIGURE 3. Parabolic Argument

We can summarize the above theorems for a unit hybrid number $\mathbf{Z}$ with the following table.

POLAR FORM	**Character**		
Type	Spacelike	Timelike	Lightlike
Elliptic	$\emptyset$	$\cos\theta+\mathbf{V}_0\sin\theta$	$\emptyset$
Hyperbolic	$\sinh\theta+\mathbf{V}_0\cosh\theta$	$\cosh\theta+\mathbf{V}_0\sinh\theta$	$a\,(1+\mathbf{V}_0)$
Parabolic	$\emptyset$	$(\epsilon+\mathbf{V}_0)\,,\epsilon=sgnS\,(\mathbf{Z})$	$\emptyset$

Logarithm of a Hybrid Number

Let $\mathbf{Z}$ be a nonlightlike hyperbolic hybrid number. So, it can be written as

$$\mathbf{Z}=k\rho\mathbf{Z}\,(\cosh\theta+\mathbf{V}\sinh\theta)$$

such that $\rho=\|\mathbf{Z}\|$ and $k\in\{1,-1,\mathbf{V},-\mathbf{V}\}$. Differentiating $\mathbf{Z}$, we obtain

$$\begin{aligned}
d\mathbf{Z}&=k\left[(\cosh\theta+\mathbf{V}\sinh\theta)\,d\rho+\rho\,(\sinh\theta+\mathbf{V}\cosh\theta)\,d\theta\right]\\
&=\frac{k\rho\,(\cosh\theta+\mathbf{V}\sinh\theta)\,d\rho}{\rho}+\mathbf{V}k\rho\,(\cosh\theta+\mathbf{V}\sinh\theta)\,d\theta\\
&=\frac{\mathbf{Z}d\rho}{\rho}+\mathbf{V}\mathbf{Z}d\theta\\
&=\mathbf{Z}\left(\frac{d\rho}{\rho}+\mathbf{V}d\theta\right).
\end{aligned}$$

And the logarithmic equation can be written afterwards

$$\int\frac{d\mathbf{Z}}{\mathbf{Z}}=\int\frac{d\rho}{\rho}+\mathbf{V}\int d\theta\Rightarrow\ln\mathbf{Z}=\ln\rho+\mathbf{V}\theta.$$

A similar relation can be achieved for elliptic hybrid numbers also. Thus, the logarithm function for elliptic and hyperbolic hybrid numbers can be defined as

$$\ln\mathbf{Z}=\ln|\mathbf{Z}|+\mathbf{V}\theta. \tag{3.2}$$

According to this equality, we have

$$\mathbf{Z}=e^{\ln|\mathbf{Z}|+\mathbf{V}\theta}=\rho e^{\mathbf{V}\theta}.$$

On the other hand, for parabolic hybrid number $\mathbf{Z}=\epsilon+\theta\mathbf{V}$, $\epsilon=sgn\,(S\,(\mathbf{Z}))$, we have

$$d\mathbf{Z}=\mathbf{V}\text{ and }\mathbf{Z}=e^{\mathbf{V}}.$$

The logarithm of parabolic hybrid numbers is not defined.

Example 4. Let's find the logarithm of the hybrid number

$$\mathbf{Z}=3+2\mathbf{i}+\varepsilon+2\mathbf{h}=\sqrt{5}\,(\cosh\theta+\mathbf{V}\sinh\theta)$$

where $\theta=\ln\sqrt{5}$ and $\mathbf{V}=\dfrac{2\mathbf{i}+1\varepsilon+2\mathbf{h}}{2}$, $\mathbf{V}^2=1$. So, using (3.2), we obtain

$$\ln\mathbf{Z}=\ln|\mathbf{Z}|+\mathbf{V}\theta=\frac{\ln\sqrt{5}}{2}\,(2+2\mathbf{i}+1\varepsilon+2\mathbf{h}).$$

Remark 3.13. The identity

$$\log(\mathbf{Z}_1\mathbf{Z}_2) = \log\mathbf{Z}_1 + \log\mathbf{Z}_2$$

which is well known for the real numbers, is not correct for the hybrid numbers, since $\mathbf{Z}_1\mathbf{Z}_2 \neq \mathbf{Z}_2\mathbf{Z}_1$.

Euler Formulas for the Hybrid Numbers

Using the serial expansions of exponential, hyperbolic and trigonometric functions, we can express the Euler formulas of unit hybrid numbers as follows.

$$\begin{array}{ll} \mathbf{Z} = e^{\mathbf{V}\theta} = \cos\theta + \mathbf{V}\sin\theta, & \mathbf{Z} \text{ is elliptic} \\ \mathbf{Z} = e^{\mathbf{V}\theta} = \cosh\theta + \mathbf{V}\sinh\theta & \mathbf{Z} \text{ is timelike hyperbolic} \\ \mathbf{Z} = \mathbf{V}e^{\mathbf{V}\theta} = \sinh\theta + \mathbf{V}\cosh\theta & \mathbf{Z} \text{ is spacelike hyperbolic} \\ \mathbf{Z} = e^{\mathbf{V}\theta} = \epsilon + \mathbf{V}\theta,\ \epsilon = sgn\,(S\,(\mathbf{Z})) & \mathbf{Z} \text{ is parabolic} \end{array}$$

Notice that these formulas are the generalized forms of the Euler formulas well known for complex, dual and hyperbolic numbers.

4. De Moivre Formula for Hybrid Numbers

Theorem 4.1. *Let* $\mathbf{Z} = x + \mathbf{V}y$, $\mathbf{V}^2 \in \{\pm 1, 0\}$ *be a nonlightlike hybrid number. Then,*

$$\begin{cases} \mathbf{Z}^n = \rho^n(\cos n\theta + \mathbf{V}\sin n\theta) & \textit{If } \mathbf{Z} \textit{ is elliptic} \\ \mathbf{Z}^n = k^n\rho^n(\cosh n\theta + \mathbf{V}\sinh n\theta) & \textit{If } \mathbf{Z} \textit{ is hyperbolic} \\ \mathbf{Z}^n = \rho^n\left(\epsilon^n + n\epsilon^{n-1}\mathbf{V}\right) & \textit{If } \mathbf{Z} \textit{ is parabolic} \end{cases}$$

where $\theta = \arg\mathbf{Z}$, $\epsilon = sgn\,(S\,(\mathbf{Z}))$, $\rho = \|\mathbf{Z}\|$ *and* $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$.

If $\mathbf{Z} = a\,(1 + \mathbf{V})$ *is a lightlike hybrid number, then,*

$$\mathbf{Z}^n = a^n 2^{n-1}(1 + \mathbf{V}).$$

Proof. We prove it by induction. Let $\mathbf{Z}$ be a nonlightlike hybrid number. If $\mathbf{Z}$ is a hyperbolic hybrid number, we have

$$\mathbf{Z} = k\rho(\cosh\theta + \mathbf{V}\sinh\theta)$$

where $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$ and $\mathbf{V}^2 = 1$. Assume that

$$\mathbf{Z}^n = k^n\rho^n(\cosh n\theta + \mathbf{V}\sinh n\theta)$$

holds. Then, we get

$$\begin{aligned} \mathbf{Z}^{n+1} &= \mathbf{Z}^n\mathbf{Z} = k\rho(\cosh\theta + \mathbf{V}\sinh\theta)\,k^n\rho^n(\cosh n\theta + \mathbf{V}\sinh n\theta) \\ &= k^{n+1}\rho^{n+1}(\cosh((n+1)\theta) + \mathbf{V}\sinh((n+1)\theta)). \end{aligned}$$

Hence, the formula is true. Moreover, since $\mathbf{Z}^{-1} = \rho^{-1}k^{-1}(\cosh\theta - \mathbf{V}\sinh\theta)$, we can write,

$$\begin{aligned} \mathbf{Z}^{-n} &= k^{-n}\rho^{-n}(\cosh(n\theta) - \mathbf{V}\sinh(n\theta)) \\ &= k^{-n}\rho^{-n}(\cosh(-n\theta) + \mathbf{V}\sinh(-n\theta)). \end{aligned}$$

So, the formula holds for all integers. It can be similarly proved for elliptical and parabolic numbers.

In the case $\mathbf{Z}$ is a lightlike hybrid number, assume that $\mathbf{Z}^n = a^n 2^{n-1}(1+\mathbf{V})$ is true for $\mathbf{Z} = a(1+\mathbf{V})$. Then,

$$\begin{aligned}
\mathbf{Z}^{n+1} &= a^n 2^{n-1}(1+\mathbf{V})\, a(1+\mathbf{V}) \\
&= a^{n+1} 2^{n-1}(1+\mathbf{V})^2 \\
&= a^{n+1} 2^n (1+\mathbf{V}).
\end{aligned}$$

□

Example 5. Let's calculate $\left(\sqrt{2}+3\mathbf{i}+\varepsilon+\mathbf{h}\right)^{10}$ and $(10+6\mathbf{i}+3\varepsilon+6\mathbf{h})^5$ using De Moivre formula..

We must first write each hybrid number in the polar form.

a) $\sqrt{2}+3\mathbf{i}+\varepsilon+\mathbf{h} = 2(\cos\frac{\pi}{4}+\frac{3\mathbf{i}+\varepsilon+\mathbf{h}}{\sqrt{2}}\frac{\pi}{4})$. $\mathbf{V}_0 = \frac{3\mathbf{i}+1\varepsilon+1\mathbf{h}}{\sqrt{2}}$, $\mathbf{V}_0^2 = -1$.

Thus, we obtain

$$\begin{aligned}
\left(\sqrt{2}+3\mathbf{i}+\varepsilon+\mathbf{h}\right)^{10} &= 2^{10}(\cos\frac{5\pi}{2}+\frac{3\mathbf{i}+\varepsilon+\mathbf{h}}{\sqrt{2}}\frac{5\pi}{2}) \\
&= 2^9\sqrt{2}\,(3\mathbf{i}+\varepsilon+\mathbf{h}).
\end{aligned}$$

b) $10+6\mathbf{i}+3\varepsilon+6\mathbf{h} = 2^3\left(\cosh(\ln 2)+\frac{2\mathbf{i}+1\varepsilon+2\mathbf{h}}{2}\sinh(\ln 2)\right)$,

$\mathbf{V}_0 = \frac{2\mathbf{i}+1\varepsilon+2\mathbf{h}}{2}$, $\mathbf{V}_0^2 = 1$. And we find

$$\begin{aligned}
(10+6\mathbf{i}+3\varepsilon+6\mathbf{h})^5 &= 2^{15}\left(\cosh(5\ln 2)+\frac{2\mathbf{i}+1\varepsilon+2\mathbf{h}}{2}\sinh(5\ln 2)\right) \\
&= 2^{15}\left(\frac{2^5+2^{-5}}{2}+\frac{2\mathbf{i}+1\varepsilon+2\mathbf{h}}{2}\frac{2^5-2^{-5}}{2}\right) \\
&= 2^8\left(2^{11}+2+\left(2^{10}-1\right)(2\mathbf{i}+1\varepsilon+2\mathbf{h})\right).
\end{aligned}$$

5. Roots of The Hybrid Numbers

In this section, the roots of a hybrid number will be examined by similar methods, as in the quaternions and the split quaternions. Some formulas will be given to find the roots according to the type of the hybrid number, as it is in the split quaternions [38]. Let $\mathbf{W} \in \mathbb{K}$ and $n \in \mathbb{Z}^+$, the hybrid numbers satisfying the equation $\mathbf{Z}^n = \mathbf{W}$ is called the roots of the n-th degree of the hybrid number $\mathbf{W}$. Finding the roots of a hybrid number will vary depending on the its type (parabolic, elliptic, hyperbolic) and characters (timelike, spacelike, lightlike). So, we give the roots of a hybrid number in two separate cases. The first one is for nonlightlike hybrid numbers, and the other is for lightlike hybrid numbers.

Theorem 5.1. *Let $\mathbf{W} = a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ be a nonlightlike hybrid number, then the roots of n-th degree of the hybrid number $\mathbf{W}$ are as follows.*

1. *If* $\mathbf{W} = \rho(\cos\theta + \mathbf{V}\sin\theta)$ *is an elliptic hybrid number, there are* n *roots and these are*

$$\mathbf{Z}_m = \sqrt[n]{\rho}\left(\cos\frac{\theta+2m\pi}{n} + \mathbf{V}\sin\frac{\theta+2m\pi}{n}\right) \tag{5.1}$$

where $m = 0, 1, 2, ..., n-1$.

2. *If* $\mathbf{W} = \rho k(\cosh\theta + \mathbf{V}\sinh\theta)$ *is a hyperbolic hybrid number, then roots of* $\mathbf{W}$ *are in the form of*

$$\mathbf{Z}_k = \begin{cases} \sqrt[n]{\rho}\left(\cosh\dfrac{\theta}{n} + \mathbf{V}\sinh\dfrac{\theta}{n}\right) & n \text{ is odd} \\ k\sqrt[n]{\rho}\left(\cosh\dfrac{\theta}{n} + \mathbf{V}\sinh\dfrac{\theta}{n}\right) & n \text{ is even and } \mathbf{W} \text{ is timelike with } a > 0 \\ \text{No roots} & \text{other cases} \end{cases}$$

where $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$.

3. *If* $\mathbf{W} = \rho(\epsilon + \mathbf{V})$, $\epsilon = sgn(S(\mathbf{Z}))$ *is a parabolic hybrid number, the roots of* W *are as follows.*

$$\mathbf{Z} = \begin{cases} \sqrt[n]{\rho}\left(\epsilon + \dfrac{\mathbf{V}}{n}\right) & n \text{ is odd} \\ \pm\sqrt[n]{\epsilon\rho}\left(\epsilon + \dfrac{\mathbf{V}}{n}\right) & n \text{ is even and } \epsilon > 0 \\ \text{No roots} & n \text{ is even and } \epsilon < 0 \end{cases}$$

Proof. **1.** If $\mathbf{W} = \rho(\cos\theta + \mathbf{V}\sin\theta)$ is an elliptic hybrid number, then we have $\mathbf{V}^2 = -1$. Let

$$\mathbf{Z} = \rho_{\mathbf{z}}(\cos\alpha + \mathbf{V}\sin\alpha)$$

be a root of the equation $\mathbf{Z}^n = \mathbf{W}$. Then, using Theorem (4.1), we have

$$\mathbf{Z}^n = \rho_{\mathbf{z}}^n(\cos n\alpha + \mathbf{V}\sin n\alpha) = \rho(\cos\theta + \mathbf{V}\sin\theta).$$

Thus, we obtain the roots (5.1).

2. If $\mathbf{W} = \rho k(\cosh\theta + \mathbf{V}\sinh\theta)$ is a nonlightlike hyperbolic hybrid number, then $\mathbf{V}^2 = 1$. Let

$$\mathbf{Z} = k_{\mathbf{z}}\rho_{\mathbf{z}}(\cosh\alpha + \mathbf{V}\sinh\alpha), \; k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$$

be a root of the equation $\mathbf{Z}^n = \mathbf{W}$. Then, using Theorem (4.1), we have

$$\mathbf{Z}^n = k_{\mathbf{z}}^n\rho_{\mathbf{z}}^n(\cosh n\alpha + \mathbf{V}\sinh n\alpha) = \rho k(\cosh\theta + \mathbf{V}\sinh\theta).$$

Let's examine this equality whether n is odd or even.
i) If n is an odd number, $k_{\mathbf{z}}^n = k_{\mathbf{z}}$, since $\mathbf{V}^2 = 1$. In this case, the equality

$$k_{\mathbf{z}}\rho_{\mathbf{z}}^n(\cosh n\alpha + \mathbf{V}\sinh n\alpha) = \rho k(\cosh\theta + \mathbf{V}\sinh\theta)$$

is satisfied if and only if $k_{\mathbf{z}} = k$. It means that the character of n-th root of a hybrid number is the same as the character of this hybrid number. So, the only root is

$$\mathbf{Z} = \sqrt[n]{\rho}\left(\cosh\frac{\theta}{n} + \mathbf{V}\sinh\frac{\theta}{n}\right).$$

ii) If n is an even number, $k_{\mathbf{Z}}^n = 1$. In this case, we have

$$\rho_{\mathbf{Z}}^n (\cosh n\alpha + \mathbf{V}\sinh n\alpha) = \rho k (\cosh\theta + \mathbf{V}\sinh\theta).$$

In order to obtain a solution for this equality, k must be 1. It means that a hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ has an n-th degree root if and only if $a > 0$ and $\mathbf{Z}$ is timelike. Therefore, we get

$$\rho_{\mathbf{Z}}^n = \rho \text{ and } n\alpha = \theta.$$

So, n-th degree of the roots of the hybrid number $\mathbf{W}$ become

$$\mathbf{Z} = k\sqrt[n]{\rho}\left(\cosh\frac{\theta}{n} + \mathbf{V}\sinh\frac{\theta}{n}\right),\ k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}.$$

That is, there are four roots if n is an even number.

3. If $\mathbf{W} = \rho(\epsilon + \mathbf{V})$, $\epsilon = sgn(S(\mathbf{Z}))$ is a parabolic hybrid number, then $\mathbf{V}^2 = 0$. Let

$$\mathbf{Z} = \rho_{\mathbf{Z}}(\epsilon + \lambda\mathbf{V})$$

be a root of the equation $\mathbf{Z}^n = \mathbf{W}$. Then, from the equalities,

$$\mathbf{Z}^n = \rho_{\mathbf{Z}}^n\left(\epsilon^n + \epsilon^{n-1} n\lambda\mathbf{V}\right) = \rho(\epsilon + \mathbf{V})$$

we find that

$$\rho_{\mathbf{Z}}^n\epsilon^n = \rho\epsilon \text{ and } \epsilon^{n-1} n\lambda\rho_{\mathbf{Z}}^n = \rho.$$

Therefore,

$$\rho_{\mathbf{Z}} = \sqrt[n]{\epsilon^{n-1}\rho} \text{ and } \lambda = \frac{1}{n}.$$

If n is an odd number, then the root is

$$\mathbf{Z} = \sqrt[n]{\rho}\left(\epsilon + \frac{\mathbf{V}}{n}\right).$$

If n is an even number, then the root is

$$\mathbf{Z} = \pm\sqrt[n]{\epsilon\rho}\left(\epsilon + \frac{\mathbf{V}}{n}\right).$$

So, in the case n is an even number, a parabolic hybrid number has a root if and only if the scalar part of the hybrid number is positive. □

Example 6. Let's find $\sqrt{\mathbf{W}}$ and $\sqrt[3]{\mathbf{W}}$ for $\mathbf{W} = 3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h}$. Since $\mathbf{W}$ is a timelike hyperbolic hybrid number, it can be written as

$$\mathbf{W} = \sqrt{5}(\cosh\theta + \mathbf{V}\sinh\theta)$$

where $\cosh\theta = \frac{3}{\sqrt{5}}$, $\sinh\theta = \frac{2}{\sqrt{5}}$ and $\mathbf{V} = \frac{2\mathbf{i} + \varepsilon + 2\mathbf{h}}{2}$, $\mathbf{V}^2 = 1$. So, we obtain

$$\mathbf{Z}_k = \sqrt{\mathbf{W}} = k\sqrt[4]{5}\left(\cosh\frac{\theta}{2} + \mathbf{V}\sinh\frac{\theta}{2}\right) = \frac{k}{2}\left(\sqrt{5} + 1 + (\sqrt{5} - 1)\mathbf{V}\right)$$

for $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$. Similarly, it can be found that

$$\sqrt[3]{\mathbf{W}} = \sqrt[6]{5}\left(\cosh\frac{\theta}{3} + \mathbf{V}\sinh\frac{\theta}{3}\right) = \frac{1}{2}\left(\sqrt[3]{5} + 1 + (\sqrt[3]{5} - 1)\mathbf{V}\right).$$

Example 7. Let's find $\sqrt{\mathbf{W}}$ and $\sqrt[3]{\mathbf{W}}$ for the timelike parabolic hybrid number $\mathbf{W} = 2 + 8\mathbf{i} + 3\varepsilon + 4\mathbf{h} = \mathbf{2}\,(1 + \mathbf{V})$ where $\mathbf{V} = \dfrac{8\mathbf{i} + 3\varepsilon + 4\mathbf{h}}{2}$, $\mathbf{V}^2 = 0$. Using the above theorem, we get

$$\begin{aligned}\sqrt{\mathbf{W}} &= \pm\sqrt{2}\left(1 + \frac{8\mathbf{i} + 3\varepsilon + 4\mathbf{h}}{4}\right),\\ \sqrt[3]{\mathbf{W}} &= \sqrt[3]{2}\left(1 + \frac{8\mathbf{i} + 3\varepsilon + 4\mathbf{h}}{6}\right).\end{aligned}$$

Theorem 5.2. *Let $\mathbf{W} = a\,(1 + \mathbf{V})$ is a lightlike hybrid number, then the roots of n-th degree of the hybrid number $\mathbf{W}$ is*

$$\mathbf{Z} = \begin{cases} \dfrac{\pm\sqrt[n]{2a}}{2}\,(1 + \mathbf{V}) & n \text{ even} \\[2ex] \dfrac{\sqrt[n]{2a}}{2}\,(1 + \mathbf{V}) & n \text{ odd} \end{cases}.$$

Proof. If $\mathbf{W}$ is a lightlike hyperbolic hybrid number, then it is in the form $\mathbf{W} = a\,(1 + \mathbf{V})$, $\mathbf{V}^2 = 1$. Let

$$\mathbf{Z} = c\,(1 + \mathbf{V})$$

be a root of the equation $\mathbf{Z}^n = \mathbf{W}$. Then, using the Theorem (4.1), we have,

$$\mathbf{Z}^n = c^n 2^{n-1}\,(1 + \mathbf{V}).$$

Then, from the equality

$$a\,(1 + \mathbf{V}) = c^n 2^{n-1}\,(1 + \mathbf{V}),$$

we obtain

$$c = \frac{\pm\sqrt[n]{2a}}{2}.$$

So, we find $c = \dfrac{\pm\sqrt[n]{2a}}{2}$ or $c = \dfrac{\sqrt[n]{2a}}{2}$ depending on whether the number n is odd or even, respectively. $\square$

Example 8. Let's find the square root of the lightlike hybrid number $\mathbf{W} = 3 + 2\mathbf{i} + \varepsilon + 3\mathbf{h}$. It can be written as

$$\mathbf{W} = 3\,(1 + \mathbf{V})$$

where $\mathbf{V} = \dfrac{2\mathbf{i} + 1\varepsilon + 3\mathbf{h}}{3}$, $\mathbf{V}^2 = 1$. So, using the above Theorem, we find

$$\mathbf{Z}_1 = \frac{\sqrt{6}}{2}\left(1 + \frac{2\mathbf{i} + 1\varepsilon + 3\mathbf{h}}{3}\right) \text{ and } \mathbf{Z}_1 = \frac{-\sqrt{6}}{2}\left(1 + \frac{2\mathbf{i} + 1\varepsilon + 3\mathbf{h}}{3}\right).$$

Thus, we get two lightlike roots.

5.1. Finding The Roots of a Hyperbolic Hybrid Number Using Null Frame Hyperbolic numbers can also be expressed with a base $\{\mathbf{e}_1, \mathbf{e}_2\}$ that provides the conditions $\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = \mathbf{e}_1\mathbf{e}_2 = 0$, $\mathbf{e}_1^2 = \mathbf{e}_1$ and $\mathbf{e}_2^2 = \mathbf{e}_2$ called the idempotent base [23], [31]. Similarly, we can express the hyperbolic hybrid numbers according to the idempotent base. It is much easier to find the roots of a hyperbolic hybrid number according to this base. Thus, roots can be obtained without using the de Moivre formula. Let us consider the numbers,

$$\mathbf{e}_1 = \frac{1-\mathbf{V}}{2} \text{ and } \mathbf{e}_2 = \frac{1+\mathbf{V}}{2}$$

in the set of hyperbolic hybrid numbers where $\mathbf{V}^2 = 1$. These two null hybrid numbers are orthogonal to each other. That is, $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy the equalities,

$$\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = \mathbf{e}_1\mathbf{e}_2 = 0.$$

Each hyperbolic hybrid number $\mathbf{Z} = x + \mathbf{V}y$ can be written as the linear combination

$$\mathbf{Z} = (x-y)\,\mathbf{e}_1 + (x+y)\,\mathbf{e}_2$$

of these two numbers $\mathbf{e}_1$ and $\mathbf{e}_2$. Shortly, we can represent a hyperbolic hybrid number by

$$\mathbf{z} = z_-\mathbf{e}_1 + z_+\mathbf{e}_2$$

where $z_- = x - y$, $z_+ = x + y$.

Definition 5.3. The set

$$\mathcal{I} = \left\{\mathbf{e}_1 = \frac{1-\mathbf{V}}{2}, \mathbf{e}_2 = \frac{1+\mathbf{V}}{2}\right\}$$

is called a null base or idempotent base for the set of hyperbolic hybrid numbers in the hybridian plane spanning by 1 and $\mathbf{V}$. They are also referred to as the idempotent base because they provide the equalities

$$\mathbf{e}_1^2 = \mathbf{e}_1 \text{ and } \mathbf{e}_2^2 = \mathbf{e}_2.$$

We will call null the coordinates of a hyperbolic hybrid number $\mathbf{Z} = x + \mathbf{V}y$ relative to the base $\{\mathbf{e}_1, \mathbf{e}_2\}$ as the null coordinates of $\mathbf{Z}$. According to this, the null coordinates of $\mathbf{Z} = x + \mathbf{V}y$ can be written as $\mathbf{Z}_n = (z_-, z_+)$.

For example, the null coordinates of the hybrid number $\mathbf{Z} = 3+2\mathbf{i}+\varepsilon+2\mathbf{h}$ is $\mathbf{Z} = (1, 5)$ since, $\mathbf{Z} = 3 + 2\mathbf{V}$ where, $\mathbf{V} = (2\mathbf{i} + \varepsilon + 2\mathbf{h})/2$.

Conversely, each hybrid number in the form $\mathbf{Z} = z_-\mathbf{e}_1 + z_+\mathbf{e}_2$ can be written as

$$\mathbf{Z} = \left(\frac{z_+ + z_-}{2}, \frac{z_+ - z_-}{2}\right) = \frac{z_+ + z_-}{2} + \frac{z_+ - z_-}{2}\mathbf{V}.$$

where $\mathbf{V} = \mathbf{e}_2 - \mathbf{e}_1$.

The addition and the multiplication operations for the hyperbolic hybrid numbers can be defined by using null coordinates as follows. Let $\mathbf{Z}_n = (z_-, z_+)$ and $\mathbf{W}_n = (w_-, w_+)$ two hyperbolic hybrid numbers lying in the hybridian plane spanned by $\{1, \mathbf{V}\}$. The equality of two hyperbolic numbers is defined as

$$\begin{aligned}
\mathbf{Z}_n = \mathbf{W}_n \quad &\Leftrightarrow \quad (z_-, z_+) = (w_-, w_+) \Leftrightarrow z_- = w_- \text{ and } z_+ = w_+ \\
\mathbf{Z}_n + \mathbf{W}_n \quad &= \quad (z_-, z_+) + (w_-, w_+) = (z_- + w_-, z_+ + w_+) ; \\
\mathbf{Z}_n \mathbf{W}_n \quad &= \quad (z_-, z_+) \cdot (w_-, w_+) = (z_- w_-, z_+ w_+) .
\end{aligned}$$

Accordingly, we can express the following theorem and corollaries. This theorem and corollaries are consistent with the Theorems (5.1) and (5.2).

Theorem 5.4. *Let $\mathbf{Z} = x + \mathbf{h}y$ be a hyperbolic hybrid number having the null coordinates $\mathbf{Z}_n = (z_-, z_+)$. Then the followings are satisfied.*
i. The conjugate of $\mathbf{Z}_n$ is $\overline{\mathbf{Z}}_n = (z_+, z_-)$.
ii. $\mathbf{Z}\overline{\mathbf{Z}} = z_- z_+$.
iii. The norm of $\mathbf{Z}_n$ is $\|\mathbf{Z}_n\| = \|(z_-, z_+)\| = \sqrt{|z_- \cdot z_+|}$.
iv. The inverse of a non null hyperbolic hybrid number $\mathbf{Z}_n$ is $\mathbf{Z}_n^{-1} = \left(z_-^{-1}, z_+^{-1}\right)$.
v. $\mathbf{Z}_n^m$ is equal to $\left(z_+^m, z_-^m\right)$ or $k\left(z_+^m, z_-^m\right)$, depending on whether the number m is odd or even, respectively, where $m \in \mathbb{Z}$ and $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$.

Proof. The proof of the first three is clear. Let $\mathbf{Z}_n$ be a non null hyperbolic hybrid number, then, $\mathbf{z}_+ \neq 0$ and $\mathbf{z}_- \neq 0$. So, we have

$$\mathbf{Z}^{-1} = \frac{1}{\mathbf{Z}_n} = \frac{\mathbf{Z}_n}{\mathbf{Z}_n \overline{\mathbf{Z}}_n} = \frac{(z_+, z_-)}{z_- z_+} = \left(\frac{1}{z_-}, \frac{1}{z_+}\right) = \left(z_-^{-1}, z_+^{-1}\right).$$

v. is the direct result of the multiplication process for the hybrid numbers. $\square$

Corollary 5.5. *Let $\mathbf{Z} = x + y\mathbf{V}$ be a hyperbolic hybrid number carrying the null coordinates $\mathbf{Z}_n = (z_-, z_+)$. Then, the roots of m-th degree of $\mathbf{Z}$ are,*

$$\mathbf{Z}_n^{1/m} = \begin{cases} k\left(\dfrac{z_+^{1/m} + z_-^{1/m}}{2} + \dfrac{z_+^{1/k} - z_-^{1/m}}{2}\mathbf{V}\right) & m \text{ is even} \\[2ex] \dfrac{z_+^{1/m} + z_-^{1/m}}{2} + \dfrac{z_+^{1/k} - z_-^{1/m}}{2}\mathbf{V} & m \text{ is odd} \end{cases}$$

such that $m \in \mathbb{Z}^+$ and $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$.

Remark 5.6. Square root of a hyperbolic hybrid number in the form $\mathbf{W} = x + y\mathbf{V}$ exists if and only if $x - y > 0$ and $x > 0$. It means that $\mathbf{W}$ must be a timelike hyperbolic hybrid number.

Example 9. Let's find $\sqrt{\mathbf{W}}$ and $\sqrt[3]{\mathbf{W}}$ for $\mathbf{W} = 13 - 8\mathbf{i} - 12\varepsilon - 4\mathbf{h}$ using the Corollary (5.5). $\mathbf{W}$ is a hyperbolic hybrid number and can be written as $\mathbf{W} = 13 - 12\mathbf{V}$, where $\mathbf{V} = \dfrac{2\mathbf{i} + 3\varepsilon + \mathbf{h}}{3}$. Therefore, we find

$$\sqrt{\mathbf{W}} = \sqrt{13 - 12\mathbf{V}} = \left(25^{1/2}, 1^{1/2}\right) = k\left(\frac{1+5}{2} + \frac{1-5}{2}\mathbf{V}\right) = k\,(3 - 2\mathbf{V}),$$

where $k \in \{1, -1, \mathbf{V}, -\mathbf{V}\}$ .So, the roots are :

$$3 - \frac{4}{3}\mathbf{i} - 2\varepsilon - \frac{2}{3}\mathbf{h}, \; -3 + \frac{4}{3}\mathbf{i} + 2\varepsilon + \frac{2}{3}\mathbf{h}, \; -2 + 2\mathbf{i} + 3\varepsilon + \mathbf{h}, \; 2 - 2\mathbf{i} - 3\varepsilon - \mathbf{h}.$$

Similarly, we find

$$\begin{aligned}\sqrt[3]{\mathbf{W}} &= \sqrt[3]{13-12\mathbf{V}} = \left(25^{1/3},1^{1/3}\right) = \left(\frac{1+\sqrt[3]{25}}{2}+\frac{1-\sqrt[3]{25}}{2}\mathbf{V}\right). \\ &= \frac{\sqrt[3]{25}+1}{2}+\left(\frac{1-\sqrt[3]{25}}{3}\right)\mathbf{i}+\left(\frac{1-\sqrt[3]{25}}{2}\right)\varepsilon+\left(\frac{1-\sqrt[3]{25}}{6}\right)\mathbf{h}.\end{aligned}$$

6. The Classification of 2×2 Matrices According To Hybrid Numbers

In the third section, as a conclusion of Theorem 3.1, we had shown that the classification of hybrid numbers depends entirely on the determinant and the trace of the 2×2 corresponding matrix. That is, we can classify a hybrid number $\mathbf{Z}$, with respect to the kind of the corresponding matrix $\varphi(\mathbf{Z}) = A$. Since

$$\mathcal{C}(\mathbf{Z}) = \det A \text{ and } \mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = \frac{(trA)^2-4\det A}{4}.$$

are satisfied, the type and the character of $\mathbf{Z}$ depends on the determinant and the trace of the matrix A. Therefore, the hybrid number corresponding to A is spacelike, timelike or lightlike if $\det A<0$, $\det A>0$, $\det A=0$ respectively. By the way, the eigenvalues and the eigenvectors of the matrix $\varphi(\mathbf{Z})$, corresponding to the hybrid number $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ are

$$\lambda = a\pm\sqrt{\mathcal{C}_{\mathcal{E}}(\mathbf{Z})} \text{ and } \mathbf{v}=\left(c+\sqrt{\mathcal{C}_{\mathcal{E}}(\mathbf{Z})},c-b+d\right).$$

So, the eigenvalues and the eigenvectors of the matrix $\varphi(\mathbf{Z})$ depend on the equality $\triangle_A=(trA)^2-4\det A$ which is equal to discriminant of the characteristic polynomial of a 2×2 real matrix A. The eigenvalues of the matrix A are complex numbers, real numbers or equals, with respect to $\triangle_A<0$, $\triangle_A>0$, $\triangle_A=0$ respectively. According to this, the hybrid number corresponding to the matrix A is elliptic, hyperbolic or parabolic if $\triangle_A<0$, $\triangle_A>0$, $\triangle_A=0$ respectively.

Just as we classify a hybrid number based on the trace and determinant of the corresponding matrix, a 2×2 matrix can also be named using the classifications as in the hybrid numbers. Accordingly, we can name any 2×2 matrix as spacelike, timelike or lightlike, or hyperbolic, elliptic or parabolic, taking into account the isomorphisms and relations between hybrid numbers and 2×2 matrices. The following definitions can be used for this.

Definition 6.1. Let A be a 2×2 real valued matrix. Then, A is spacelike, timelike or lightlike if $\det A<0$, $\det A>0$, $\det A=0$ respectively.

Definition 6.2. Let A be a 2×2 real valued matrix where λ_1 and λ_2 are the eigenvalues of A. We call that A is hyperbolic, elliptic or parabolic matrix if λ_1,λ_2 are real numbers, λ_1,λ_2 are complex numbers or $\lambda_1=\lambda_2$ respectively.

Instead of the definition (6.2), the following definition can be given.

Definition 6.3. Let A be a 2×2 real valued matrix. Then, A is hyperbolic, elliptic or parabolic matrix if $\triangle_A > 0$, $\triangle_A < 0$,or $\triangle_A = 0$ respectively, where $\triangle_A = (trA)^2 - 4\det A$ is the discriminant of the characteristic polynomial of A.

Any 2×2 matrix can be classified with the following table.

A	$\det A > 0$	$\det A = 0$	$\det A < 0$
$(\text{trA})^2 < 4\text{detA}$	Timelike Elliptic	$\emptyset$	$\emptyset$
$(\text{trA})^2 = 4\text{detA}$	Timelike Parabolic	Null Parabolic	$\emptyset$
$(\text{trA})^2 > 4\text{detA}$	Timelike Hyperbolic	Null Hyperbolic	Spacelike Hyperbolic

Example 10. The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix}$$

are real numbers. It means that A is a hyperbolic matrix. Also, it can be seen that A is a spacelike hyperbolic matrix, since $\det A < 0$ and $(trA)^2 > 4\det A$. This matrix corresponds to the spacelike hyperbolic hybrid number $\mathbf{Z} = \frac{5}{2} - \frac{7}{2}\mathbf{i} - \frac{3}{2}\varepsilon + 5\mathbf{h}$.

Example 11. The only eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & 9 \\ -1 & -1 \end{bmatrix}$$

is 2. That is, $\lambda_1 = \lambda_2 = 2$. So A is a timelike parabolic matrix, since $\det A > 0$. Also, the matrix A corresponds to the timelike parabolic hybrid number $\mathbf{Z} = 2 + 8\mathbf{i} + 3\varepsilon + 4\mathbf{h}$.

Notice that the rotation matrices in the Euclidean, Lorentzian and the Galilean plane,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \begin{bmatrix} \theta+1 & -\theta \\ \theta & 1-\theta \end{bmatrix}$$

are elliptic, hyperbolic, and parabolic matrices where these matrices correspond to elliptic, hyperbolic and parabolic hybrid numbers respectively.

Using these classifications and De Moivre formulas for hybrid numbers, the roots of a 2×2 matrix can be found easily. The study on this subject will be examined in the next article [40].

Now let's give a simple linear algebra application of hybrid numbers.

Corollary 6.4. *A 2×2 matrix, having only one eigenspace* $\mathbf{Sp}\{\overrightarrow{\mathbf{u}} = (x, y)\}$ *associated with the eigenvalue λ can be generated as follows*

$$T(\lambda, x, y) = \begin{bmatrix} x+\lambda & -x^2/y \\ y & \lambda - x \end{bmatrix}. \tag{6.1}$$

Proof. Since this eigenspace is spanned by only one vector, the eigenvalues are equal to each other. Therefore, the eigenvalue and the eigenvector should be $\lambda = a$ and $\mathbf{v} = (c, c - b + d) = (x, y)$ respectively. Each parabolic hybrid number in the form

$$\mathbf{Z} = \lambda + b\mathbf{i} + x\varepsilon + (y + b - x)\,\mathbf{h},$$

generates the matrix desired. From the equality $\mathcal{C}_{\mathcal{E}}(\mathbf{Z}) = 0$, we have

$$-b^2 + 2bx + (y + b - x)^2 = (x - y)^2 + 2by = 0.$$

Thus, we get $b = \dfrac{(x-y)^2}{-2y}$. So, we find

$$\mathbf{Z} = \lambda + \frac{(x-y)^2}{-2y}\mathbf{i} + x\varepsilon + \left(\frac{x^2 - y^2}{-2y}\right)\mathbf{h}.$$

And the corresponding matrix is (6.1). □

Example 12. Let's find a matrix such that the eigenspace associated with $\lambda = 7$ equals $\mathrm{Sp}\left\{\overrightarrow{\mathbf{u}} = (4, 11)\right\}$ is the only eigenspace. Using (6.1), we obtain

$$T(7, 4, 11) = \begin{bmatrix} 11 & -16/11 \\ 11 & 3 \end{bmatrix}.$$

Conclusions

In this work, we have examined a new ring of numbers, which is non-commutative and has the unit element. We named this number set as hybrid numbers because it is a linear combination of well-known complex, hyperbolic and dual numbers. We have given the relation $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$ between the units $\{\mathbf{i}, \varepsilon, \mathbf{h}\}$ of these three number systems, and we have seen the algebraic and geometric consistency of this relation. Thus, we have obtained a non-commutative algebra which unified all three number systems. We can briefly summarize the results obtained in this study as follows.

1. There are some relations between the dual, complex and the hyperbolic numbers. And this relations can be given with matrix representations.

2. Any hybrid number can be similar to one of the complex, dual or hyperbolic numbers. This fact also confirms that any two-dimensional unitary algebra is isomorphic to either dual, hyperbolic, or complex numbers, since the set of hybrid numbers is a two-dimensional commutative algebra spanned by 1 and $b\mathbf{i} + c\varepsilon + d\mathbf{h}$.

3. With the classifications of the hybrid numbers, they can be expressed via the polar representations. Therefore, we proved the De Moivre formula for the hybrid numbers considering the classifications.

4. Roots of a hybrid number were founded with some formulas which are proved.

5. A 2 by 2 matrix can be classified using the hybrid numbers.

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