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On New Formulas of Fibonacci and Lucas Numbers Involving Golden Ratio Associated with Atomic Structure in Chemistry

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Abstract: The main purpose of this paper is to give many new formulas involving the Fibonacci numbers, the golden ratio, the Lucas numbers, and other special numbers. By using generating functions for the special numbers with their functional equations method, we also give many new relations among the Fibonacci numbers, the Lucas numbers, the golden ratio, the Stirling numbers, and other special numbers. Moreover, some applications of the Fibonacci numbers and the golden ratio in chemistry are given.

Keywords: Fibonacci and Lucas numbers; golden ratio; generating functions; Stirling numbers; periodic table

MSC: 05A10; 05A15; 11B37; 11B39; 11B73

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1. Introduction

In almost all the intellectual achievements of humankind, the discovery of different sets of numbers has played many important roles. These discoveries led to the development of mathematical analysis and number theory. The applications of different number sets are perhaps among the most important inventions of humanity. Therefore, with the aid the different sets of numbers, applications of analytical number theory and mathematical analysis have been given in different fields of almost all sciences. For example, the wonderful synergy between the Fibonacci numbers and the golden ratio, thanks to the discovery of different sets of numbers, is incredibly integrated with all fields of science, especially chemistry. It is well known that the Fibonacci numbers and other special numbers are also used in mathematical models that frequently appear in theory of chemistry theory involving the structure of the elements, the periodic table of the elements, and their symmetrical and crystalline structures (cf. [1–16]).

By using an approximation method for the Fibonacci numbers, Wlodarski [15,16] have studied the world of atoms and the proton–neutron ratio in the nucleus. He gave many results between the proton–neutron ratio and the golden ratio. Boeyens et al. [3,4]) have studied chemistry and its application using number theory involving the Fibonacci numbers, the Lucas numbers, and the golden ratio. They gave some basic aspects of the golden helix related to the simulation of chemical events. They concluded that molecular mechanics is an ideal method for structure optimization based on parameters obtained by number theory (cf. [3,4]).

The motivation of this paper is to provide many new formulas and relations involving the Fibonacci numbers, the Lucas numbers, the golden ratio, the Stirling numbers, and other special numbers with the aid of the generating functions for the special numbers . We give some remarks and observations for the Fibonacci numbers and the golden ratio in chemistry. The results of this paper, including those involving the Fibonacci numbers, the

Lucas numbers, other special numbers and the golden ratio, perhaps have the potential to be used in new mathematical modeling, besides finding the position of new elements in the periodic system.

In this paper the following notations and definitions are used.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of positive integers, the set of integers, the set of rational numbers, and the set of real numbers, respectively. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $x \in \mathbb{R}$. The falling factorial polynomials $x_{(n)}$ is given by

$$x_{(n)} = x(x-1)(x-2)\dots(x-n+1),$$

where $x_{(0)} = 1$ and $n \in \mathbb{N}$.

The Stirling numbers of the first kind $S_1(v,d)$ are defined by

$$y_{(v)} = \sum_{d=0}^{v} S_1(v,d) y^d \tag{1}$$

(cf. [14]). Note that $(-1)^{v-d} S_1(v,d)$ denotes the signed Stirling numbers of the first kind.

The Stirling numbers of the second kind $S_2(v,d)$ are defined by

$$\sum_{n=0}^{\infty} S_2(n,d) \frac{w^n}{n!} = \frac{(e^w-1)^d}{d!} \tag{2}$$

and

$$y^v = \sum_{d=0}^{v} S_2(v,d) y_{(d)} \tag{3}$$

(cf. [14]).

The Fibonacci numbers are defined by the following generating function:

$$G_F(w) = \sum_{n=0}^{\infty} F_n w^n = \frac{w}{1-w-w^2} \tag{4}$$

(cf. [7]).

The Lucas numbers are defined by the following generating function:

$$\sum_{n=0}^{\infty} L_n w^n = \frac{2-w}{1-w-w^2} \tag{5}$$

(cf. [7]).

There are many applications of the generating functions for the Fibonacci numbers and the Lucas numbers (cf. [3–14]).

When the denominator of the function $G_F(w)$ is equal to 0, one easily gets the following poles of this function:

$$1-w-w^2 = (1-\alpha w)(1-\beta w) = 0,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}$$

which is the golden ratio and

$$\beta = \frac{1-\sqrt{5}}{2}.$$

If $|w| < \min\left\{\frac{1}{\alpha}, \frac{1}{-\beta}\right\}$, the series in (4) is convergent (cf. [7]).

The Binet formulas for the numbers F_n and L_n are given as follows, respectively:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{6}$$

and

$$L_n = \alpha^n + \beta^n \tag{7}$$

(cf. [7]).

The well-known relations among the numbers F_n, L_n, α and β are given as follows:

$$\begin{aligned} \alpha^n &= \alpha F_n + F_{n-1}, \\ \beta^n &= \beta F_n + F_{n-1}, \\ \alpha + \beta &= 1, \\ \alpha - \beta &= \sqrt{5}, \end{aligned}$$

and

$$\alpha\beta = -1$$

(cf. [7]).

Replacing n by $2n$ in (6), we have

$$\begin{aligned} F_{2n} &= \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)(\alpha^n + \beta^n) \\ &= F_n L_n \end{aligned}$$

(cf. [7]).

By the help of the Binet formulas, the numbers F_n and L_n are also defined by the following exponential generating functions, respectively:

$$\sum_{n=0}^{\infty} F_n \frac{w^n}{n!} = \frac{e^{\alpha w} - e^{\beta w}}{\alpha - \beta} \tag{8}$$

and

$$\sum_{n=0}^{\infty} L_n \frac{w^n}{n!} = e^{\alpha w} + e^{\beta w} \tag{9}$$

(cf. [3–14]).

By using (2) and (8), we have

$$\sum_{n=0}^{\infty} F_n \frac{w^n}{n!} = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \beta^n \frac{w^n}{n!} \sum_{n=0}^{\infty} (\alpha - \beta)^n S_2(n,1) \frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} F_n \frac{w^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \beta^{n-j} (\alpha - \beta)^{j-1} S_2(j,1) \frac{w^n}{n!}.$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get

$$F_n = \sum_{j=0}^{n} \binom{n}{j} \left(F_{n-j-1} + \beta F_{n-j}\right) \sqrt{5^{j-1}} S_2(j,1).$$

Since a set of j elements can only be partitioned in a single way into 1 or j subsets, one has $S_2(j,1) = 1$. Thus, we have

$$F_n = \sum_{j=0}^{n} \binom{n}{j} \left(F_{n-j-1} + \beta F_{n-j}\right) \sqrt{5^{j-1}}.$$

By using (9), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(F_{j-1}+\beta F_j\right)L_{n-j}\frac{w^n}{n!}=\sum_{n=0}^{\infty}(\alpha-\beta)^n\frac{w^n}{n!}+1.$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, for $n \geq 1$, we have

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(F_{j-1}+\beta F_j\right)L_{n-j}=\sqrt{5^n}.$$

The numbers $\ell_n(v) := \ell_n((0,1,2,\ldots,v-1))$ are defined by the following generating function:

$$G_1(w;v)=\frac{w^v}{\prod_{j=0}^{v-1}(e^w-j)}=\sum_{n=0}^{\infty}\ell_n(v)\frac{w^n}{n!}, \tag{10}$$

where

$$\ell_n(v) := \ell_n((0,1,2,\ldots,v-1)),$$

for $m \in \mathbb{N}$,

$$\ell_0(v)=0, \ell_0(m+1)=\ell_1(m+1)=\cdots=\ell_{m-1}(m+1)=0$$

and

$$\ell_m(m+1)\neq 0$$

(cf. [12]).

By using (10), we get the following equation:

$$w^v=\prod_{j=0}^{v-1}(e^w-j)\sum_{n=0}^{\infty}\ell_n(v)\frac{w^n}{n!}. \tag{11}$$

Substituting $v = 1$ into (11), we get

$$w=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\ell_j(1)\frac{w^n}{n!}.$$

Therefore

$$\sum_{j=0}^{n}\binom{n}{j}\ell_j(1)=\begin{cases}0 & if\ n\neq 1\\ 1 & if\ n=1.\end{cases}$$

With the mathematical induction method, the following result is reached:

$$\begin{aligned} w^v &= e^w(e^w-1)(e^w-2)\cdots(e^w-v+1)\sum_{n=0}^{\infty}\ell_n(v)\frac{w^n}{n!} \\ &= \sum_{m=0}^{v}S_1(v,m)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}m^j\ell_{n-j}(v)\frac{w^n}{n!}. \end{aligned} \tag{12}$$

Thus,

$$\sum_{m=0}^{v}S_1(v,m)\sum_{j=0}^{n}\binom{n}{j}m^j\ell_{n-j}(v)=\begin{cases}0 & if\ n\neq v\\ n! & if\ n=v\end{cases}$$

(cf. [12]).

Setting

$$\sum_{m=0}^{v}S_1(v,m)m^j=P(j,v),$$

therefore, we have the following novel formula

$$\sum_{j=0}^{n}\binom{n}{j}P(j,v)\ell_{n-j}(v)=\begin{cases}0 & if\ n\neq v\\ n! & if\ n=v.\end{cases}$$

The results obtained in this article are briefly summarized as follows:

In Section 2, we give many new formulas involving the numbers F_n, L_n, α, β, and the Stirling numbers.

In Section 3, using genearting functions with their functional equations, we give many new identities and relations involving the numbers F_n, L_n, $\ell_n(v)$, golden ratio α, and also combinatorial sums.

In Section 4, we give some aspects of the golden ratio α application in chemistry.

In Section 5, we give a conclusion section.

2. Relations and Identities among the Numbers F_n, L_n, α and β

In this section, we give interesting relations and formulas between the numbers F_n, L_n, α (golden ratio), and β. These results have the potential to be used in number theory, chemistry, biology, physics, and other areas.

Theorem 1. *Let $k\in\mathbb{N}_0$. Then we have*

$$\alpha^k=\frac{L_k+(\alpha-\beta)F_k}{2}. \tag{13}$$

Proof. We need the following well-known equations (cf. [10]):

$$H(\alpha x;z)+H(\beta x;z)=\sum_{k=0}^{\infty}\left(\begin{array}{c}k+z\\k\end{array}\right)L_kx^k \tag{14}$$

and

$$H(\alpha x;z)-H(\beta x;z)=(\alpha-\beta)\sum_{k=0}^{\infty}\left(\begin{array}{c}k+z\\k\end{array}\right)F_kx^k, \tag{15}$$

where the function $H(x;z)$ is given by

$$H(x;z)=\sum_{k=0}^{\infty}\left(\begin{array}{c}z+k\\k\end{array}\right)x^k, \tag{16}$$

where $|x|<1$. Using (14) and (15), we get

$$2\sum_{k=0}^{\infty}\left(\begin{array}{c}z+k\\k\end{array}\right)(\alpha x)^k=\sum_{k=0}^{\infty}\left(\begin{array}{c}k+z\\k\end{array}\right)L_kx^k+(\alpha-\beta)\sum_{k=0}^{\infty}\left(\begin{array}{c}k+z\\k\end{array}\right)F_kx^k.$$

Comparing the coefficients of $\left(\begin{array}{c}k+z\\k\end{array}\right)x^k$ on both sides of the above equation, we get

$$2\alpha^k=L_k+\sqrt{5}F_k.$$

Thus, the proof of theorem is completed. □

Substituting $x=\alpha y$ into (16), we get

$$H(\alpha y;z)=\sum_{k=0}^{\infty}\left(\begin{array}{c}z+k\\k\end{array}\right)(\alpha y)^k.$$

Combining the above equation with the well-known Chu–Vandermonde identity

$$\binom{z+a}{k} = \sum_{j=0}^{k} \binom{z}{j} \binom{a}{k-j}$$

(cf. [8,14]), we get

$$\sum_{k=0}^{\infty} \binom{z+k}{k} (\alpha y)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{z}{j} \binom{k}{k-j} (F_{k-1} + \alpha F_k) y^k.$$

Combining the above equation with (1), we obtain

$$\sum_{k=0}^{\infty} \binom{z+k}{k} (\alpha y)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{d=0}^{j} \binom{k}{k-j} \frac{F_{k-1} + \alpha F_k}{j!} S_1(j,d) y^k z^d.$$

Comparing the coefficients of y^k on both sides of the above equation, we arrive at the following theorem:

Theorem 2. *Let $k \in \mathbb{N}_0$. Then we have*

$$\beta^{-k} = \frac{(-1)^k}{\binom{z+k}{k}} \sum_{j=0}^{k} \sum_{d=0}^{j} \binom{k}{k-j} \frac{F_{k-1} + \alpha F_k}{j!} S_1(j,d) z^d.$$

Theorem 3. *Let $k \in \mathbb{N}_0$. Then we have*

$$\beta^k = \frac{(\beta - \alpha) F_k + L_k}{2}.$$

Proof. Using (14) and (15), we get

$$-2 \sum_{k=0}^{\infty} \binom{z+k}{k} (\beta x)^k = -\sum_{k=0}^{\infty} \binom{k+z}{k} L_k x^k + (\alpha - \beta) \sum_{k=0}^{\infty} \binom{k+z}{k} F_k x^k.$$

Comparing the coefficients of $\binom{k+z}{k} x^k$ on both sides of the above equation, we get

$$-2\beta^k = (\alpha - \beta) F_k - L_k.$$

Thus proof of theorem is completed. □

Theorem 4. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{k} \left((-1)^{2k-3j} \binom{k}{j} F_{k-2j-1} + \alpha F_{2j-k} \right) = 1.$$

Proof. Replacing x by $x(\alpha + \beta)$ in (16), we get

$$H(x(\alpha+\beta); z) = \sum_{k=0}^{\infty} \binom{z+k}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{2j-k} x^k.$$

Since $\alpha + \beta = 1$, after some calculations in the above equation, we get

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\alpha^{2j-k} = 1.$$

Combining the above equation with the following well-known formula

$$F_{-k} = (-1)^{k+1}F_k,$$

we arrive at the desired result. □

3. Identities and Relations Involving the Numbers F_n, L_n, $\ell_n(v)$, and Golden Ratio α

In this paper, with the aid of generating functions, we give some new identities and relations involving the numbers F_n, L_n, $\ell_n(v)$, and golden ratio α.

Replacing w by αw and βw into (10), we get

$$G_1(\alpha w; v) = \sum_{n=0}^{\infty}\alpha^n\ell_n(v)\frac{w^n}{n!}$$

and

$$G_1(\beta w; v) = \sum_{n=0}^{\infty}\beta^n\ell_n(v)\frac{w^n}{n!}.$$

Combining the above equations with (6) and (7), we get

$$\frac{G_1(\alpha w; v) - G_1(\beta w; v)}{\alpha - \beta} = \sum_{n=0}^{\infty}F_n\ell_n(v)\frac{w^n}{n!}$$

and

$$G_1(\alpha w; v) + G_1(\beta w; v) = \sum_{n=0}^{\infty}L_n\ell_n(v)\frac{w^n}{n!}.$$

After some elementary calculations in the above equation, we get

$$\sum_{n=0}^{\infty}\alpha^n\ell_n(v)\frac{w^n}{n!} = \sum_{n=0}^{\infty}\frac{(\alpha-\beta)F_n + L_n}{2}\ell_n(v)\frac{w^n}{n!}.$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at Equation (13).

Replacing w by αw in (10), we have

$$(\alpha w)^v = \prod_{j=0}^{v-1}(e^{\alpha w} - j)\sum_{n=0}^{\infty}\ell_n(v)\frac{(\alpha w)^n}{n!}.$$

Combining the above equation with (12), we obtain

$$(\alpha w)^v = \sum_{m=0}^{v}S_1(v,m)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\alpha^n m^j\ell_{n-j}(v)\frac{w^n}{n!}.$$

Comparing the coefficients of w^n on both sides of the above equation, we get

$$\sum_{m=0}^{v}S_1(v,m)\sum_{j=0}^{n}\binom{n}{j}\alpha^n m^j\ell_{n-j}(v) = \begin{cases} 0 & if\ n \neq v \\ n!\alpha^v & if\ n = v. \end{cases} \quad (17)$$

Replacing w to βw in (10), we have

$$(\beta w)^v = \prod_{j=0}^{v-1}\left(e^{\beta w} - j\right) \sum_{n=0}^{\infty} \ell_n(v) \frac{(\beta w)^n}{n!}$$

Combining the above equation with (12), we obtain

$$(\beta w)^v = \sum_{m=0}^{v} S_1(v,m) \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \beta^n m^j \ell_{n-j}(v) \frac{w^n}{n!}.$$

Comparing the coefficients of w^n on both sides of the above equation, we get

$$\sum_{m=0}^{v} S_1(v,m) \sum_{j=0}^{n} \binom{n}{j} \beta^n m^j \ell_{n-j}(v) = \begin{cases} 0 & if\ n \neq v \\ n!\beta^v & if\ n = v. \end{cases} \tag{18}$$

After adding Equations (17) and (18) side by side and performing some calculations, with the aid of Equation (7), we arrive at the following theorem:

Theorem 5. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{v} S_1(v,m) \sum_{j=0}^{n} \binom{n}{j} m^j \ell_{n-j}(v) L_n = \begin{cases} 0 & if\ n \neq v \\ n!L_v & if\ n = v. \end{cases}$$

Corollary 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} P(j,v) \ell_{n-j}(v) L_n = \begin{cases} 0 & if\ n \neq v \\ n!L_v & if\ n = v. \end{cases}$$

After subtracting Equation (17) from Equation (18) side by side, dividing both sides of the equation by $\alpha - \beta$ and performing the necessary operations, with the aid of Equation (6), the following relation is obtained:

Theorem 6. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{v} S_1(v,m) \sum_{j=0}^{n} \binom{n}{j} m^j \ell_{n-j}(v) F_n = \begin{cases} 0 & if\ n \neq v \\ n!F_v & if\ n = v. \end{cases}$$

Corollary 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} P(j,v) \ell_{n-j}(v) F_n = \begin{cases} 0 & if\ n \neq v \\ n!F_v & if\ n = v. \end{cases}$$

4. Some Aspects of the Application of the Golden Ratio α in Chemistry

Many new and interesting formulas containing the golden ratio have been given in the previous sections. At this point, we pose the following question: What are the uses and applications of these new formulas in other fields, especially in chemistry?

To answer this question, in this section, we try to explain some well-known applications of the golden ratio to the theory of chemistry. To help with the study and investigation of the answer, we give the following comments and remarks.

In the work of Wlodarski [16], there are four fundamental asymmetries in the world of atoms. These appear in the structure of atomic nuclei of protons and neutrons, in the distribution of fission fragments by mass number resulting from the bombardment of most heavy nuclei by thermal neutrons, in the distribution of a number of isotopes of even stable elements, and also in the distribution of emitted particles in two opposite directions in weak nuclear interactions. Consequently, the numerical values of all these asymmetries are approximately equal to the golden ratio. These numerical values are also related to the

Fibonacci numbers. The number of protons Z in the lightest stable nucleus as a rule is equal the number of neutrons N. When the atomic number Z increases, the proton–neutron ratio in the nucleus $\frac{Z}{N}$ decreases to about 0.6. One can easily see that symmetrical fission of most heavy nuclei by slow neutrons is very rare. The number of protons and neutrons of fission fragments in the nuclear reaction is one of the Fibonacci numbers (cf. [15,16]).

It is well known that there are many applications of the golden ratio to periodicity and periodic table. Therefore, in the work of Boeyens and Levendis ([4], p. 259), they suggest that, due to their digital binary make-up, a valid physical model of neutral atoms, and of matter in general, follows directly from number theory. The numerical basis of nuclide periodicity is given by in Figure 8.1 in the work of Boeyens and Levendis's ([4], p. 260). The point of convergence appears to be similar in Figure 6.1, which is given in ([4], p. 210), which shows the variation of the proton:neutron ratio with atomic (or mass) number. If the two points of convergence are identical, then it means that

$$\frac{N-Z}{Z} = \frac{Z}{N}. \tag{19}$$

The limiting of the above ratio $\frac{Z}{N}$ for stable nuclides is equal to the golden ratio α. That is,

$$Z = \alpha N$$

(cf. [3,4]). Due to Equation (19), it well-known that the golden ratio occurs in periodic table. Furthermore, atoms in the quasicrystalline structure follow the golden ratio. The golden ratio also appears in atomic physics.

Observe that the simple equation given by (19) derived from the above ratio is of course well known to correspond to the ratio obtained by any point C at a distance x from point B on the line segment $|AB|$ of unit length. That is,

$$\frac{|AC|}{|CB|} = \frac{|CB|}{|AB|} \Longrightarrow \frac{1-x}{x} = \frac{x}{1}.$$

With the help of the above equation, the golden ratio α is defined by the irrational number $\alpha = x = \frac{1+\sqrt{5}}{2}$ (cf. [3–5,7,15]).

Different forms of Equation (19) have also been studied by many researchers. A relation between the golden ratio and atomic structure has been studied using this ratio. In the work of Currie [5], we also see that the chart of nuclides is correctly understood by the golden ratio.

It is seen that a relation between the atomic numbers of the elements in the periodic system is given by the golden ratio. In addition, this ratio gives relations among the number of protons, the number of neutrons, and the number of electrons. With this ratio, it is possible to predict the place and some fundamental properties of the elements that are still undiscovered in the periodic system. Considering the ongoing research today, it is a known fact that the periodic system has not been completed yet and that new elements will be added. Consequently, thanks to mathematical modeling involving the golden ratio, we can find some important information about possible new elements that are found. When considered mathematically, assumptions can be made in the investigation of the place and some fundamental properties of new elements in the periodic system, which can be well defined using the Fibonacci numbers in related models.

By using generating functions and well-known formulas involving the Fibonacci numbers and the Lucas numbers, we derive some new formulas related to the golden ratio, which is given by the Equation (13). Combining the golden ratio with this formula and the other formulas, we may get links between Z, N, and the limiting of the ratio $\frac{Z}{N}$.

5. Conclusions

In this paper, we introduced many new formulas, relations, and combinatorial sums involving the Fibonacci numbers, the Lucas numbers, the golden ratio, the Stirling numbers, special numbers, and the Chu–Vandermonde convolution formula. Many applications of the golden ratio with the Fibonacchi numbers related to the theory of chemistry have been given. The relations between these formulas and theory of chemistry were discussed in terms of the following question:

What are the uses and applications of these new formulas in other fields, especially in chemistry? We suggest the following directions for future investigations:

- We will investigate a partial answer to the above question;
- Using the results of this paper, new mathematical models, which are related to the golden ratio and also the structure of atomic nuclei of protons and neutrons, will be investigated.

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