Non-Lightlike Darboux Curves in Minkowski $n-$Space

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Abstract

G. Saban studied in [4] on Darboux curves and obtained some results about Darboux curves on a surfaces in $\mathbb{E}^3$. Ergin examined this results in [1] for a timelike curve on timelike surface in Minkowski 3 space $\mathbb{E}_1^3$ and defined generalized Darboux curve on a hypersurface M in $\mathbb{E}^n$ in [2]. In this paper we defined nonlightlike Darboux curve on a hypersurface M in $\mathbb{E}_1^n$ and gave similar theorems to those in [1] and [2].

Keywords : Darboux Curve, Timelike, Spacelike Curve

1 Introduction

The Minkowski space $\mathbb{E}_1^n$ is the Euclidean space $\mathbb{E}^n$ provided with the Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_1 v_1 + \sum_{i=2}^{n} u_i v_i$$

where $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$.

An arbitrary vector be $\mathbf{u} = (u_1, u_2, ..., u_n)$ in $\mathbb{E}_1^n$ can have one of three Lorentzian causal characters; it is spacelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, it is timelike if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ or it is null (or lightlike) if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^n$ is locally spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$.

The pseudo-norm of an arbitrary vector $\mathbf{u} \in \mathbb{E}_1^n$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}.$$

The sphere of center $C = (m_1, m_2, ..., m_n)$ and radius $r \in \mathbb{R}^+$ in the Minkowski n-space is defined by

$$S(r) = \{\mathbf{u} = (u_1, u_2, ..., u_n) \in \mathbb{E}_1^n : \langle \mathbf{u} - C, \mathbf{u} - C \rangle = \pm r^2\}.$$

Let $M \subset \mathbb{E}_1^n$ be a nonlightlike curve with the coordinate neighborhood (I, α) and parameterized by arclenght s. In this paper, we always assume that the derivatives of α order 1,2,...,n are linearly independent at any point. Suppose that $\{\mathbf{V}_1(s), \mathbf{V}_2(s), ..., \mathbf{V}_n(s)\}$ is frenet frame and $\kappa_i(s)$ is i-th curvature function at the point $\alpha(s)$. Then Frenet formulas are found as follows

$\mathbf{V}_1'(s) = \epsilon_2 \kappa_1(s).\mathbf{V}_2(s)$
$\mathbf{V}_i'(s) = -\epsilon_{i-1}\kappa_{i-1}(s).\mathbf{V}_{i-1}(s) + \epsilon_{i+1}\kappa_i(s).\mathbf{V}_{i+1}(s)$, $2 \leq i \leq n-1$
$\mathbf{V}_n'(s) = -\epsilon_{n-1}\kappa_{n-1}(s).\mathbf{V}_{n-1}(s)$
where $\epsilon_i = \langle \mathbf{V}_i, \mathbf{V}_i \rangle$.

And also, for a regular nonlightlike curve $\alpha : I \to \mathbb{E}_1^n$ whose Frenet n-frame is $\{\mathbf{V}_1(s), \mathbf{V}_2(s), ...\mathbf{V}_n(s)\}$ at the point $\alpha(s)$, we have

$$\alpha^{(p)} = \sum_{j=1}^{p} a_j \mathbf{V}_j, \;\; 1 \leq p \leq n.$$

where $a_j \in \mathbb{R}$. It can be proved by using induction process on p. [6].

2 Osculating p-spheres in Minkowski n-space

Definition 2.1 *We say that α and $F^{-1}(0)$ have k-point contact for $t = t_0$ provided the function g defined by*

$$g(t) = F(\alpha_1(t), \alpha_2(t), ..., \alpha_n(t)) = F(\alpha(t))$$

satisfies $g(t_0) = g'(t_0) = ... = g^{(k-1)}(t_0) = 0$, $g^{(k)} \neq 0$. We also say that the order of contact is k. Dropping the condition $g^{(k)} \neq 0$ we say that there is at least k-point contact.

This definition means in the former language of geometers that the curve α and $F^{-1}(0)$ "meet at k consecutive points" or that α and $F^{-1}(0)$ "meet at k infinitely close points".

Definition 2.2 *For $k = 1, 2, .., n-1$ an osculating k-sphere at a point of a nonlightlike curve in the Lorentzian space $\mathbb{E}_1^n$ is a k-dimensional sphere having at least $(k+2)-$point contact with the curve at that point. For $k = n-1$ we will simply write osculating hypersphere.*

Definition 2.3 *The focal curve of a smooth curve $\alpha : I \to \mathbb{E}^n$ in the Euclidean space $\mathbb{E}^n$ consist of the centers its osculating hyperspheres. The centers of the osculating hypersphere of α are well defined only for the points of α where all curvatures are different from zero. Let $\alpha : I \to E_1^n$ be a nonlightlike curve having all its curvature different from zero at any point. The curve $C_\alpha : \theta \to C_\alpha(\theta) \in \mathbb{E}_1^n$ consisting of the centers of the osculating hyperspheres of the curve $\alpha : \theta \to \alpha(\theta) \in E_1^n$ is called the parametrized focal curve of α. The focal curve can be write as*

$$C_\alpha(\theta) = \left(\alpha + c_1\mathbf{V}_2 + c_2\mathbf{V}_3 + ... + c_{(n-1)}\mathbf{V}_n\right)(\theta)$$

where $c_1, c_2, ..., c_{(n-1)}$ are smooth functions of the parameter of the curve α [5]. For instance, for a spacelike curve with timelike binormal $\mathbf{B}$ in $\mathbb{E}_1^3$, we have

$$C_\alpha(\theta) = \alpha(\theta) + \frac{1}{\kappa(\theta)}\mathbf{N}(\theta) + \frac{-\kappa'(\theta)}{\kappa^2(\theta)\tau(\theta)}\mathbf{B}(\theta)$$

where $\kappa(\theta)$, $\tau(\theta)$ are the curvature and torsion functions and $\mathbf{N}(\theta)$, $\mathbf{B}(\theta)$ are the normal and binormal respectively of α at $\alpha(\theta)$.

Definition 2.4 *We call the function c_i, the $i-th$ focal curvature of α.*

Let $\alpha : I \to \mathbb{E}_1^n$ be a spacelike curve whose curvatures $\kappa_1, \kappa_2, ...\kappa_{n-1}$ are different from zero at any point and whose Frenet frame is $\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_n$.

Theorem 2.1 *The focal curvatures of the nonlightlike curve $\alpha : I \to \mathbb{E}_1^n$, parametrized by arc lenght s, satisfy the following equations for $c_{(n-1)} \neq 0$*

$$\begin{cases} 1 = \epsilon_1 c_1 \kappa_1, \\ c_1' = \epsilon_2 c_2 \kappa_2, \\ c_i' + c_{(i-1)}\kappa_i\epsilon_i - \epsilon_i c_{(i+1)}\kappa_{(i+1)} = 0, \qquad \text{for } 1 \leq i \leq n-2 \\ \epsilon_n c_{(n-1)}' - \dfrac{\left(R_{(n-1)}^2\right)'}{2c_{(n-1)}} = -c_{(n-2)}\kappa_{(n-1)}. \end{cases} . \tag{2.1}$$

Before the proof of this theorem, we need to prove the following lemma.

Lemma 2.1. *Let $\alpha : s \to (\alpha_1(s), \alpha_2(s), ..., \alpha_n(s))$ be a nonlightlike curve in $\mathbb{E}_1^n$. The velocity vector $C'(s)$ of the focal curve of α at s is proportional to the last normal vector $V_n(s)$ of α.*

Proof : Consider the family of functions $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$F_q(s) = \frac{1}{2} \|C - \alpha(s)\|^2.$$

We can write $-F = \langle \alpha, C \rangle - g - \dfrac{C^2}{2}$, where $g = \dfrac{\langle \alpha, \alpha \rangle}{2}$. Here, the following system of n equations defines the focal curve $C(s)$ of α.

$$\left.\begin{array}{r} \langle \alpha', C(s) \rangle - g' = 0 \\ \langle \alpha'', C(s) \rangle - g'' = 0 \\ \vdots \\ \left\langle \alpha^{(n)}, C(s) \right\rangle - g^{(n)} = 0 \end{array}\right\} \tag{2.2}$$

Differentiating each equation with respect to θ, we obtains the following system of equations:

$$\left.\begin{array}{r} \langle \alpha', C'(s) \rangle + \langle \alpha'', C(s) \rangle - g'' = 0 \\ \langle \alpha'', C'(s) \rangle + \langle \alpha''', C(s) \rangle - g''' = 0 \\ \vdots \\ \left\langle \alpha^{(n+1)}, C'(s) \right\rangle + \left\langle \alpha^{(n+1)}, C(s) \right\rangle - g^{(n+1)} = 0 \end{array}\right\} \tag{2.3}$$

Combining the *i-th* equation of system (2.3) with the $(i+1)$-*th* equation of system (2.2), for $i = 1, 2, ..., n-1$, we obtain

$$\left.\begin{array}{r} \langle \alpha', C'(s) \rangle = 0 \\ \langle \alpha'', C'(s) \rangle = 0 \\ \vdots \\ \left\langle \alpha^{(n)}, C'(s) \right\rangle = 0 \end{array}\right\} \tag{2.4}$$

This means that the velocity vector of the focal curve is orthogonal to osculating hyperplane of α. Namely, $C'(s)$ is proportional to the last normal vector V_n. ■

Proof of Theorem 2.1 : Let $C_\alpha(s) = \left(\alpha + c_1\mathbf{V}_2 + c_2\mathbf{V}_3 + ... + c_{(n-1)}\mathbf{V}_n\right)(s)$ be the focal curve of α. Applying Frenet equations to the derivative of C_α with respect to the arc lenght of α, we obtain:

$$\begin{aligned} C'_\alpha(s) &= \alpha' + c'_1\mathbf{V}_2 + c_1\left(-\epsilon_1\kappa_1\mathbf{V}_1 + \epsilon_3\kappa_2\mathbf{V}_3\right) + c'_2\mathbf{V}_3 + \\ &\quad + c_2\left(-\epsilon_2\kappa_2\mathbf{V}_2 + \epsilon_4\kappa_3\mathbf{V}_4\right) + ... + c'_{(n-2)}\mathbf{V}_{(n-1)} + \\ &\quad + c_{(n-2)}\left(-\epsilon_{(n-2)}\kappa_{(n-2)}\mathbf{V}_{(n-2)} + \epsilon_n\kappa_{(n-1)}\mathbf{V}_n\right) \\ &\quad + c'_{(n-1)}\mathbf{V}_n + c_{(n-1)}\left(-\epsilon_{n-1}\kappa_{n-1}\mathbf{V}_{n-1}\right) \\ C'_\alpha(s) &= \mathbf{V}_1\left(1 - \epsilon_1 c_1\kappa_1\right) + \mathbf{V}_2\left(c'_1 - c_2\epsilon_2\kappa_2\right) + \\ &\quad + \mathbf{V}_3\left(c'_2 + c_1\epsilon_3\kappa_2 - \epsilon_3 c_3\kappa_3\right) + \\ &\quad + \mathbf{V}_i\left(c'_i + c_{(i-1)}\kappa_i\epsilon_i - \epsilon_i c_{(i+1)}\kappa_{(i+1)}\right) + \\ &\quad + ... + \mathbf{V}_n\left(c'_{(n-1)} + \epsilon_n c_{(n-2)}\kappa_{(n-1)}\right). \end{aligned}$$

Because of that $C'_\alpha(\theta)$ is proportional to the last normal vector $\mathbf{V}_n$, the first $n-1$ components of C'_α vanish. This implies that

$$\begin{aligned} 1 &= \epsilon_1 c_1\kappa_1 \\ c'_1 &= c_2\epsilon_2\kappa_2 \\ c'_2 &= \left(-c_1\kappa_2 + c_3\kappa_3\right)\epsilon_3 \\ &\vdots \\ c'_{n-2} &= \left(-c_{n-3}\kappa_{n-2} + c_{(n-1)}\kappa_{(n-1)}\right)\epsilon_{(n-1)} \end{aligned} \tag{2.5}$$

Thus, $C'_\alpha = \mathbf{V}_n\left(c'_{(n-1)} + \epsilon_n c_{(n-2)}\kappa_{(n-1)}\right)$. Using the fact that the radius $R_{(n-1)}$ of the osculating hypersphere satisfies $R^2_{(n-1)} = \|C_\alpha - \alpha\|^2$, we obtain

$$\begin{aligned} \left(R^2_{n-1}\right)' &= \langle C_\alpha - \alpha, C_\alpha - \alpha\rangle' \\ &= 2\langle C'_\alpha - V_1, C_\alpha - \alpha\rangle \\ &= 2\left\langle \mathbf{V}_n\left(c'_{(n-1)} + \epsilon_n c_{(n-2)}\kappa_{(n-1)}\right) - \mathbf{V}_1, c_1\mathbf{V}_2 + c_2\mathbf{V}_3 + \cdots + c_{(n-1)}\mathbf{V}_n\right\rangle \\ &= 2c_{(n-1)}\epsilon_n\left(c'_{(n-1)} + \epsilon_n c_{(n-2)}\kappa_{(n-1)}\right) \end{aligned}$$

Thus for

$$c_{(n-1)} \neq 0, \epsilon_n c'_{(n-1)} - \frac{\left(R^2_{(n-1)}\right)'}{2c_{(n-1)}} = -c_{(n-2)}\kappa_{(n-1)}.$$

This equation together with (2.5) proves Theorem 2.1. ■

Remark. *Since radius of osculating sphere of a spherical curve is constant, The equality* $\epsilon_n c'_{(n-1)} = -c_{(n-2)}\kappa_{(n-1)}$ *is satisfied for nonlightlike spherical curves.*

3 Lorentzian Darboux Curve of an Hypersurfaces in $\mathbb{E}_1^n$

Now let α be a nonlightlike curve on a nondegenerate hypersurface M in $\mathbb{E}_1^n$. If $\mathbf{X}_1$ is the unit tangent vector field of α and $\mathbf{N}$ is the unit normal vector field of M, then we can find the system of vector fields $\{\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_{n-1}, \mathbf{N}\}$ which is called natural frame field system of the pair of (α, M) where $Sp\{\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_{n-1}\} = T(M)$. And we know that i-th geodesic curvature function of α is

$$\kappa_{ig}(s) = \langle \mathbf{X}_i'(s), \mathbf{X}_{i+1}(s) \rangle, \qquad 1 \leq i \leq n-1$$

If we use $\mathbf{X}_n = \mathbf{N}$, then we have the following derivative formulas

$D_{\mathbf{X}_1}\mathbf{X}_1 = \mathbf{X}_1' = \epsilon_2^* \kappa_{1g}(s)\mathbf{X}_2(s) + \epsilon \mathbf{II}(\mathbf{X}_1, \mathbf{X}_1)\mathbf{N}$
$D_{\mathbf{X}_1}\mathbf{X}_i = \mathbf{X}_i' = -\epsilon_{i-1}^* \kappa_{(i-1)g}(s)\mathbf{X}_{i-1}(s) + \epsilon_{i+1}^* \kappa_{ig}(s)\mathbf{X}_{i+1}(s) + \epsilon_n^* \mathbf{II}(\mathbf{X}_1, \mathbf{X}_i)\mathbf{N}$, for $1 < i \leq n-1$
$D_{\mathbf{X}_1}\mathbf{N} = \mathbf{N}' = -\epsilon_n^* \mathbf{II}(\mathbf{X}_1, \mathbf{X}_1)\mathbf{X}_1 - \epsilon_n^* \mathbf{II}(\mathbf{X}_1, \mathbf{X}_2)\mathbf{X}_2 - ... - \epsilon_n^* \mathbf{II}(\mathbf{X}_1, \mathbf{X}_{n-1})\mathbf{X}_{n-1}$
where $\mathbf{II}$ is the second fundamental form and $\epsilon_i^* = \langle \mathbf{X}_i, \mathbf{X}_i \rangle$, $1 \leq i \leq n$.

Definition 3.1. *Let α be a nonlightlike curve on a nondegenerate hypersurface M in $\mathbb{E}_1^n$. If the tangent plane of the osculating hypersphere of α at the point $\alpha(t)$ coincides with the tangent plane at the same point $\alpha(t)$, then the curve α is called a nonlightlike Darboux curve on M.*

From this definition, we can conclude that if α is a nonlightlike Darboux curve on M whose center of osculating hypersphere is C, then $\alpha(s) - C$ and the normal $\mathbf{N}$ of the hypersurfaces at the point $\alpha(s)$ are same direction.

Theorem 3.1. *Let C be the center of osculating sphere of α at the point nonlightlike curve $\alpha(s)$. Then,*

$$\langle C - \alpha(s), \alpha'''(s) \rangle = 0.$$

Proof : If we calculate $\alpha'''(s)$ by using Frenet formulas, we find it as

$$\alpha'''(s) = -\epsilon_2 \epsilon_1 \kappa_1^2 \mathbf{V}_1 + \epsilon_2 \kappa_1' \mathbf{V}_2 + \epsilon_2 \epsilon_3 \kappa_1 \kappa_2 \mathbf{V}_3$$

So,

$$\langle C - \alpha(s), \alpha'''(s) \rangle = \left\langle \sum_{i=2}^{n} c_{i-1}\mathbf{V}_i, -\epsilon_2 \epsilon_1 \kappa_1^2 \mathbf{V}_1 + \epsilon_2 \kappa_1' \mathbf{V}_2 + \epsilon_2 \epsilon_3 \kappa_1 \kappa_2 \mathbf{V}_3 \right\rangle.$$

Thus, we obtain

$$\langle C-\alpha(s),\alpha'''(s)\rangle=\epsilon_2 c_1(s)\kappa_1'(s)+\epsilon_2 c_2(s)\kappa_1(s)\kappa_2(s).$$

Considering the equations (2.1), we get

$$\langle C-\alpha(s),\alpha'''(s)\rangle=0 \qquad \blacksquare$$

Now, we can find the vector α''' in terms of higher order geodesic curvatures, so we obtain

$$\begin{aligned}
&\alpha'=\mathbf{X}_1\\
&\alpha''=\epsilon_2^*\kappa_{1g}(s)\mathbf{X}_2(s)+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)\mathbf{N}\\
&\alpha'''=\epsilon_2^*\kappa_{1g}'(s)\mathbf{X}_2(s)+\epsilon_2^*\kappa_{1g}(s)\mathbf{X}_2'(s)+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)'N+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)\mathbf{N}'\\
&\alpha'''=\epsilon_2^*\kappa_{1g}(s)(-\epsilon_1^*\kappa_{1g}(s)\mathbf{X}_1(s)+\epsilon_3^*\kappa_{2g}(s)\mathbf{X}_3(s)+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_2)\mathbf{N})\\
&\qquad+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)'\mathbf{N}+\epsilon_n^*\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)\mathbf{N}'+\epsilon_2^*\kappa_{1g}'(s)\mathbf{X}_2(s)
\end{aligned}$$

Therefore we find

$$\langle\alpha''',\mathbf{N}\rangle=\epsilon_2^*\kappa_{1g}(s)\mathbf{II}(\mathbf{X}_1,\mathbf{X}_2)+\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)'.$$

Definition 3.2. *Let α be a nonlightlike curve on hypersurface M in $\mathbb{E}_1^n$, then we call that*

$$\epsilon_2^*\kappa_{1g}(s)\mathbf{II}(\mathbf{X}_1,\mathbf{X}_2)+\mathbf{II}(\mathbf{X}_1,\mathbf{X}_1)'$$

is generalized Lorentzian Darboux function through α and it is denoted by $\mathcal{D}$.

Theorem 3.2. *Let M be a nondegenerate hypersurface in $\mathbb{E}_1^n$ and α be a nonlightlike curve in M. If α is a Darboux curve on M then*

$$\mathcal{D}=0$$

along the ■.

Proof : Since α is a nonlightlike Darboux curve on M. From definition, we have

$$\alpha(s)-C=\lambda N.$$

Also we defined that $\mathcal{D} = \langle \alpha'''(s), \mathbf{N} \rangle$. Therefore we obtain

$$\mathcal{D} = \langle \alpha'''(s), \lambda^{-1}(\alpha(s) - C) \rangle = 0. \qquad \blacksquare$$

Theorem 3.3. *For all curves on a hyperplane and a hypersphere in* $\mathbb{E}_1^n$, $\mathcal{D} = 0$.
Proof : We know that shape operator of hyperplane and hypersphere with radius r are 0 and $\pm\frac{1}{r}I_{n-1}$ respectively.
So,

$$\mathbf{II}(\mathbf{X}_1, \mathbf{X}_1) = \langle S(\mathbf{X}_1), \mathbf{X}_1 \rangle = \begin{cases} 0, & \text{for hyperplane} \\ \pm\frac{1}{r}, & \text{for hypersphere} \end{cases}$$

Since $\mathbf{II}(\mathbf{X}_1, \mathbf{X}_1)$ is constant, we have $(\mathbf{II}(\mathbf{X}_1, \mathbf{X}_1))' = 0$.
And $\mathbf{II}(\mathbf{X}_1, \mathbf{X}_2) = 0$ because $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle = 0$.
Thus, we get

$$\mathcal{D} = 0. \qquad \blacksquare$$

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