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ORIGINAL PAPER

Combinatorial sums and generation algorithms for beta-based words and polynomials

Irem Kucukoglu[1] · **Yilmaz Simsek**[2]

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Abstract
In this paper, by virtue of the beta polynomials, we introduce and study a new special word family referred to as the beta-based words. In order to construct the beta-based words and their generation algorithm, we inspire from the definition of the beta polynomials. By making implementation in the Wolfram language, we obtain some tables for these words and their decimal equivalents. We also show that the decimal equivalents of the beta-based words are associated with the Mersenne numbers. For their visual illustrations, we also present plots designed by black-white superimposed square blocks emerging from the first few beta-based words. Moreover, we investigate some properties of the beta-based words, involving length, height, slope, symmetry and complement properties. Additionally, in this paper, we construct a generalization of harmonic numbers. We also derive some combinatorial sums related to the beta polynomials, the Bernstein polynomials and the generalized harmonic numbers.

Keywords Combinatorics on words · Beta polynomials · Beta type rational functions · Combinatorial sums · Beta function · Bernstein polynomials · Gamma function · Toeplitz matrix · Mersenne numbers · Generalized harmonic numbers · Combinatorial generation algorithm

Mathematics Subject Classification 68R15 · 05B05 · 05A15 · 11B37 · 11B83 · 15B05 · 33B15

1 Introduction

The field of combinatorics on words, which is a new field of discrete mathematics, focuses on the study of formal languages, words and strings formed by letters or symbols. Recent studies in this field have found application in many areas such as number theory, group theory, theoretical computer science dealing with automata and formal languages. With the publication of Lothaire's books [28–30], good sources of terminology and a well-defined

✉ Irem Kucukoglu
irem.kucukoglu@alanya.edu.tr

Yilmaz Simsek
ysimsek@akdeniz.edu.tr

[1] Department of Engineering Fundamental Sciences, Faculty of Engineering, Alanya Alaaddin Keykubat University, Antalya 07425, Turkey

[2] Department of Mathematics, Faculty of Science, University of Akdeniz, Antalya 07058, Turkey

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theory of the field of combinatorics on words was formed, and this field began to develop and grow further. Thus, the researchers accelerated their studies to define new classes of special words by benefiting from these inspirational sources and by revealing relationships between some word families and special polynomial families. For instance, Bényi and Nkonkobe [4] introduced and enumerated weighted α-distanced words by showing their connection with the unified Apostol-type polynomials. Moreover, in [19, 20, 23, 25], it has been shown that there are relationships between k-ary Lyndon words and some special number and polynomial sequences, involving the Apostol-type numbers and polynomials.

Polynomial classes involving Bernstein type polynomials have also found very useful applications and modeling in all areas of mathematics, especially in combinatorics and theory of computation, as well as in other fields of science. In this context, motivated by its broad perspective applications of the Bernstein type polynomials, Kucukoglu and Simsek [22] defined the Bernstein-based words and investigated their properties by algebraic, combinatorial or algorithmic approaches.

As with Bernstein type polynomials, the beta type polynomials are also related to many different classes of polynomials and special functions. Some of these are the beta function, the Euler gamma function, the Swiss-Knife polynomials, the Euler polynomials, and etc. For some applications, see [41, 43–45] and the references cited therein.

In the context of the motivations behind these relationships, the aims of this paper are to both define new classes of words based on the definition of beta polynomials and to present their features and applications with the techniques of combinatorics on words. In addition, we are also aimed to find finite sum formulas that include the beta polynomials and to show their relationships with special numbers and other special functions.

We next recall the definitions of notions that we use to derive the results of this paper.

The beta polynomials, $\mathfrak{B}_{k,n}(x)$, are defined by

$$\mathfrak{B}_{k,n}(x) = x^k(1+x)^{n-k}, \tag{1.1}$$

where $k = 0, 1, \ldots, n$ such that $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (*cf.* [41, 43–45] and see the references cited therein).

The beta type rational functions are also defined by (*cf.* [43]):

$$\mathfrak{M}_{k,n}(x) = x^k(1+x)^{n-k}; \quad (n, k \in \mathbb{N}_0) \tag{1.2}$$

whose exponential generating functions are given by

$$\mathfrak{h}_k(t, x) := \left(\frac{x}{1+x}\right)^k e^{t(1+x)} = \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^n}{n!}, \tag{1.3}$$

where $k \in \mathbb{N}_0$.

Note that

$$\mathfrak{M}_{k,n}(x) = \begin{cases} \mathfrak{B}_{k,n}(x) & \text{if } n \geq k \\ \mathfrak{b}_{k,n}(x) & \text{if } n < k \end{cases} \tag{1.4}$$

where

$$\mathfrak{b}_{k,n}(x) = \frac{x^k}{(1+x)^{k-n}}, \qquad (n \in \{0, 1, 2, \ldots, k-1\}),$$

(*cf.* [43]).

The beta function, $B(\alpha, \beta)$, is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, \mathrm{d}t, \tag{1.5}$$

where $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ (*cf.* [32], [50, p. 9, Eq.-(60)], [51]), and the relation between the beta function $B(\alpha, \beta)$ and the Euler gamma function $\Gamma(\alpha)$ is given by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.6}$$

which, replacing α and β by natural numbers n and m respectively, yields

$$\begin{aligned} B(n, m) &= \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \\ &= \frac{(n-1)!\,(m-1)!}{(n+m-1)!}, \end{aligned} \tag{1.7}$$

due to the fact that

$$\Gamma(n) = (n-1)!; \quad (n \in \mathbb{N}_0) \tag{1.8}$$

(*cf.* [32, 50, 51]).

It is well-known that

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{(2k)!}{4^k k!}\sqrt{\pi} \tag{1.9}$$

and

$$\Gamma(z+1) = z\Gamma(z) \tag{1.10}$$

(*cf.* [32, 50, 51]).

Let $t \in \mathbb{C}$, $x \in [0, 1]$ and $k \in \mathbb{N}_0$. Then, the generating function for the Bernstein basis functions, $B_k^n(x)$, is given as follows:

$$\frac{(xt)^k \, e^{(1-x)t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}, \tag{1.11}$$

such that the Bernstein basis functions are also given by the following explicit formula:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad (k = 0, 1, \ldots, n; \;\; n \in \mathbb{N}_0) \tag{1.12}$$

which have relationships with a large number of concepts including the Catalan numbers, the binomial distribution, the Poisson distribution and etc.; for details, see [1, 27, 38–40, 42] and also the cited references therein.

Here, $\binom{n}{k}$ denotes, as usual, the binomial coefficient given for $n, k \in \mathbb{N}_0$ as below:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

such that it is assumed that $\binom{n}{k} = 0$ when $k < 0$ and $k > n$.

Note that the integral formula for the Bernstein basis functions is given as follows:

$$\int_0^1 B_k^n(x) \mathrm{d}x = \binom{n}{k} B(k+1, n-k+1) \tag{1.13}$$

$$= \frac{1}{n+1}, \tag{1.14}$$

(*cf.* [42]; see also [13, Eq.-(5.28), p.254], [27]).

In [21], Kucukoglu and Simsek introduced the polynomials $\alpha_k(\lambda; n)$ by the following ordinary generating functions:

$$\mathcal{F}_\alpha(z, \lambda; n) := \frac{(\lambda(1+\lambda z)-1)^{n+1} - (\lambda-1)^{n+1}}{z\lambda^2(n+1)} = \sum_{k=0}^{\infty} \alpha_k(\lambda; n) z^k. \tag{1.15}$$

where $k \in \mathbb{N}_0$ and $\lambda, z \in \mathbb{R}\backslash\{0\}$ (or $\mathbb{C}\backslash\{0\}$) (*cf.* [21]).

An explicit formula for the polynomials $\alpha_k(\lambda; n)$ is given by

$$\alpha_k(\lambda; n) = \frac{(n)_k \lambda^{2k} (\lambda-1)^{n-k}}{(k+1)!}. \tag{1.16}$$

where $k, n \in \mathbb{N}_0$ and $(n)_k = n(n-1)\ldots(n-k+1)$ (*cf.* [21]).

In the sequel, we continue to remember the terminology of combinatorics on words, from the books of Lothaire [28–30] and Shallit [37].

Let Σ be a nonempty set called alphabet, each element of which is called a letter. A finite sequence of letters, in the following form:

$$\Omega = (a_1, a_2, \ldots, a_n), \quad \forall a_i \in \Sigma; \quad i = 1, 2, \ldots, n,$$

is called a finite word of length n over the alphabet Σ. If we use Σ^* to denote the set of all words over the alphabet Σ, then $\Omega \in \Sigma^*$.

The concatenation, or so-called juxtaposition, is a binary operation on the set Σ^* with the following mapping:

$$\Sigma^* \times \Sigma^* \mapsto \Sigma^*$$

concatenating two words as in the following way:

$$\Omega_1\Omega_2 = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m). \tag{1.17}$$

where $\Omega_1 = (a_1, a_2, \ldots, a_n) \in \Sigma^*$ and $\Omega_2 = (b_1, b_2, \ldots, b_m) \in \Sigma^*$.

The concatenation of two words is well-defined, closed and an associative binary operation which is not commutative. Moreover, the set Σ^* has an identity element ϵ denoting the empty word which is a neutral element for concatenation. Due to these features, when it is equipped with the concatenation of two words, Σ^* forms an algebraic structure called free monoid over the alphabet Σ. Therefore, a word in the form of

$$\Omega = (a_1, a_2, \ldots, a_n)$$

can be expressed as

$$\Omega = a_1 a_2 \cdots a_n \tag{1.18}$$

(*cf.* [19, 28–30, 37]).

The nth power of a word Ω is obtained by concatenating itself side by side as follows:

$$\Omega^n = \begin{cases} \underbrace{\Omega\Omega\cdots\Omega}_{n-\text{times}} & \text{if } n \in \mathbb{N}, \\ \epsilon & \text{if } n = 0, \end{cases}$$

(*cf.* [37, p. 3]). For example, $(01)^3 = 010101$.

The reversal (or so-called mirror image) of a word $\Omega = a_1 a_2 \cdots a_n$ is defined by the word

$$\Omega^R = (a_1 a_2 \cdots a_n)^R = a_n a_{n-1} \cdots a_1 \tag{1.19}$$

(*cf.* [37, p. 11], [30, p. 4]).

From [37, p. 2], it is well-known that the binary complement of a binary word Ω is obtained by interchanging $\mathtt{0}$ and $\mathtt{1}$ in Ω and denoted by $\overline{\Omega}$, which is sometimes called as logical complement, ones' complement or the Boolean complement in Boolean algebra with the notation $\neg\Omega$ (*cf.* [37, p. 2]). For example, $\neg\mathtt{0100} = \overline{\mathtt{0100}} = \mathtt{1011}$.

The slope of a nonempty word Ω is defined by (*cf.* [29]):

$$\mathtt{slope}\,(\Omega) = \frac{\mathtt{height}\,(\Omega)}{|\Omega|} \tag{1.20}$$

where $\mathtt{height}\,(\Omega)$ denotes the height of the word Ω, which corresponds to the number of letters equal to $\mathtt{1}$ in the word Ω (*cf.* [29, pp. 42–45]; see also [3]), and $|\Omega|$ denotes the length of the word Ω, which is the number of letters that forms the word Ω. Thus

$$|a_1 a_2 \cdots a_n| = n \tag{1.21}$$

(see, for details, [19, 28–30]).

For example, for the word $\Omega = \mathtt{001111}$, $|\Omega| = 6$ and $\mathtt{height}\,(\Omega) = 4$. Therefore the slope $\mathtt{slope}\,(\Omega) = \frac{2}{3}$.

Throughout the paper, let the notation Σ_k denote the finite alphabet $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{k\text{-}1}\}$ of cardinality k.

In the remainder of this paper, we obtain some results which have been summarized section by section as follows:

In Sect. 2, we give construction of beta-based words with the inspiration of the definition of the beta polynomials. After providing a few list of the beta-based words, we investigate their some properties such as length, height, slope, complement property and symmetry property. By using the combinatorial generation algorithm of the beta-based words, we give their computational implementation in the Wolfram language and provide some tables that involves these words and their decimal equivalents. We also show that the decimal equivalents of the beta-based words are associated with the Mersenne numbers. For their visual illustrations, we also present plots designed by black-white superimposed square blocks emerging from the first few beta-based words. In Sect. 3, we provide a new generalization of harmonic numbers and derive some combinatorial sums related to the beta polynomials, the Bernstein polynomials and the generalized harmonic numbers. In Sect. 4, we finalize the paper by providing conclusions and discussions on our results and their potential applications.

2 Construction of beta-based words

In this section, inspiring from the definition of the beta polynomials given in (1.1), we define a new special word family referred to as *beta-based words* as in the following definition:

Definition 2.1 Let $n, k \in \mathbb{N}_0$. Let $x \in \Sigma_2 = \{\mathtt{0}, \mathtt{1}\}$. By considering the product $\cdot$ between letters or words as their concatenation, the beta-based words $\mathfrak{B}_w\,(x; k, n)$ over the alphabet Σ_2 are defined by

$$\mathfrak{B}_w\,(x; k, n) = \begin{cases} \mathtt{0}^k \cdot \mathtt{1}^{n-k} & \text{if } x = \mathtt{0}, \\ \mathtt{1}^k \cdot \mathtt{0}^{n-k} & \text{if } x = \mathtt{1} \end{cases} \tag{2.1}$$

with $\mathfrak{B}_w(x;0,0)=\epsilon$ and $\mathfrak{B}_w(x;k,n)=\epsilon$ when $k>n$.

In the sequel, we examine a few examples to exemplify the beta-based words produced by the Definition 2.1:

$$\begin{aligned}
\mathfrak{B}_w(0;0,4) &= 1111 & \qquad \mathfrak{B}_w(1;0,4) &= 0000\\
\mathfrak{B}_w(0;1,4) &= 0111 & \qquad \mathfrak{B}_w(1;1,4) &= 1000\\
\mathfrak{B}_w(0;2,4) &= 0011 \quad \text{and} \quad & \mathfrak{B}_w(1;2,4) &= 1100\\
\mathfrak{B}_w(0;3,4) &= 0001 & \qquad \mathfrak{B}_w(1;3,4) &= 1110\\
\mathfrak{B}_w(0;4,4) &= 0000 & \qquad \mathfrak{B}_w(1;4,4) &= 1111.
\end{aligned}$$

Remark 2.2 In the literature, there are some applications of the string patterns 0^k1^{n-k}, 1^k0^{n-k} and their reversals. For instance, during the investigation of binary bubble languages, Ruskey et al. [35] used the strings 1^k0^{n-k} and $1^{n-k}0^k$ to describe the bubble poset by calling the stings as the terminal and the unique minimum element. Moreover, during the consideration of combinatorial generation of binary stings and bubble languages, Burcsi et al. [7] also provided an algorithm that uses the word 1^k0^{n-k} in order to obtain a binary string of length n and of weight k. Besides, in [34, p. 134], Ruskey handled a list that starts with the bitstring 1^k0^{n-k} and ends with the bitstring $0^{n-k}1^k$. Especially, Ruskey [34, Theorem 5.1, p. 134] showed that there is a Hamilton path in a bipartite graph $G(n,k)$ that starts at 1^k0^{n-k} and ends at $0^{n-k}1^k$ in case of when n is even and k is odd. Apart from these, Cheon et al. [9] handled establishment on the word-representability of Toeplitz graphs $G_n(1^k0^m)$ and $G_n(0^k1^m)$ for any positive integers k and m. MacMahon [31, p. 315] gave a combinatorial interpretation of the q-binomial coefficients over the set $\mathfrak{S}\left(0^{n-k}1^k\right)$ denoting the collection of all 0-1 bit strings (a sequence of binary digits) consisting of $n-k$ zeros and k ones.

2.1 Some properties of the beta-based words

Here, some properties of the words $\mathfrak{B}_w(x;k,n)$ are given.

By the following theorem, we present length, height and slope properties of the beta-based words.

Theorem 2.3 (Length, height and slope) *Let $n,k\in\mathbb{N}_0$. Then, the length, height and slope of the beta-based words $\mathfrak{B}_w(x;k,n)$ are given by*

$$|\mathfrak{B}_w(0;k,n)|=n,\quad \mathtt{height}\left(\mathfrak{B}_w(0;k,n)\right)=n-k,\quad \mathtt{slope}\left(\mathfrak{B}_w(0;k,n)\right)=\frac{n-k}{n}$$

and

$$|\mathfrak{B}_w(1;k,n)|=n,\quad \mathtt{height}\left(\mathfrak{B}_w(1;k,n)\right)=k,\quad \mathtt{slope}\left(\mathfrak{B}_w(1;k,n)\right)=\frac{k}{n}.$$

Proof The proof of Theorem 2.3 follows directly from the equations (1.20) and (1.21). □

By the following theorem, we also present complement property of the beta-based words.

Theorem 2.4 (Complement property) *Let $n,k\in\mathbb{N}_0$. Then, the beta-based words $\mathfrak{B}_w(x;k,n)$ satisfy the following complement property:*

$$\mathfrak{B}_w(1;k,n)=\overline{\mathfrak{B}_w(0;k,n)}$$

and

$$\mathfrak{B}_w(0;k,n)=\overline{\mathfrak{B}_w(1;k,n)}.$$

Proof The proof of Theorem 2.4 follows directly from the Definition 2.1 of the beta-based words. □

Theorem 2.5 (Symmetry property with respect to vertical reflection) *Let $n, k \in \mathbb{N}_0$. Then, the beta-based words $\mathfrak{B}_w\,(x; k, n)$ satisfy the following symmetry properties:*

$$\mathfrak{B}_w\,(1; k, n) = (\mathfrak{B}_w\,(0; n-k, n))^R$$

and

$$\mathfrak{B}_w\,(0; k, n) = (\mathfrak{B}_w\,(1; n-k, n))^R\,.$$

Proof The proof of Theorem 2.5 follows from the usage of (1.19) in the Definition 2.1. □

Remark 2.6 One of the applications of the symmetry property, given in the Theorem 2.5, is that a concatenation of the words

$$\mathfrak{B}_w\,(0; n-k, n) \quad \text{and} \quad \mathfrak{B}_w\,(1; k, n)$$

or

$$\mathfrak{B}_w\,(1; n-k, n) \quad \text{and} \quad \mathfrak{B}_w\,(0; k, n)$$

generates a palindrome word spelling the same backward as forward. As an example, consider the following words:

$$\mathfrak{B}_w\,(0; 3, 4) = \texttt{0001} \quad \text{and} \quad \mathfrak{B}_w\,(1; 1, 4) = \texttt{1000}.$$

Then, since they are reversal of each other, the spelling or pronunciation of any of the above forwards is the same as the spelling or pronunciation of the other backwards. By concatenating them, we get

$$\mathfrak{B}_w\,(0; 3, 4) \cdot \mathfrak{B}_w\,(1; 1, 4) = \texttt{00011000}$$

which is a palindrome word. To see other kinds of applications related to the palindrome words, the interested reader may glance at [12].

2.2 Computational implementations for the beta-based words

Here, we provide computational implementations for the beta-based words in the Wolfram language.

By implementing the Definition 2.1 in the Wolfram Language, we present a procedure `BetaBasedWords` (see: Implementation 1).

Implementation 1 The following code, involving the procedure `BetaBasedWords` written in the Wolfram Language, returns the beta-based words $\mathfrak{B}_w\,(x; k, n)$ for $x \in \Sigma_2$ and $n, k \in \mathbb{N}_0$.

```
BetaBasedWords[x_?StringQ, 0, 0] := "\[Epsilon]"
BetaBasedWords[x_?StringQ, k_?IntegerQ, n_?IntegerQ]/; k < 0||n < 0:="undefined"
BetaBasedWords[x_?StringQ, k_?IntegerQ, n_?IntegerQ]/; k > n  := "\[Epsilon]"
BetaBasedWords[x_?StringQ, k_?IntegerQ, n_?IntegerQ]/; n >= k := First[{
   Factor1CaseZero = "" <> Table["0", {j, 1, k}];
   Factor2CaseZero = "" <> Table["1", {j, 1, n−k}];
   Factor1CaseOne = "" <> Table["1", {j, 1, k}];
   Factor2CaseOne = "" <> Table["0", {j, 1, n−k}];
   Which[x == "0", result = Factor1CaseZero <> Factor2CaseZero, x=="1", result = Factor1CaseOne
      <> Factor2CaseOne]}]
```

By executing the procedure `BetaBasedWords` in the Wolfram Mathematica (version 12.0), we also provide some values of the beta-based words as tables obtained just for a few special cases (among others). The entries of Tables 1 and 2 show some of the beta-based words $\mathfrak{B}_w(x;k,n)$, in the special case when $x = 0$ and $x = 1$ with $n \in \{0,1,\ldots,8\}$ and $k \in \{0,1,\ldots,6\}$. Note that the entries ϵ of Tables 1 and 2 denote the empty word.

If we convert the beta-based words $\mathfrak{B}_w(x;k,n)$ to decimal numbers, we obtain the numbers provided in Tables 3 and 4 for the corresponding special cases.

Remark 2.7 Recall that numbers of the form

$$M_n = 2^n - 1; \quad (n \in \mathbb{N}_0)$$

are called Mersenne numbers and these numbers form a sequence corresponding to the sequence `A000225` in the Sloane's On-Line Encyclopedia of Integer Sequences (OEIS). See, for details, [46, OEIS: `A000225`] and [52]. Observe that each column of Table 3 contains the Mersenne numbers, and thus the decimal equivalents of the beta-based words $\mathfrak{B}_w(x;k,n)$ are corresponding to the Mersenne numbers. Even, when we look at which words have decimal equivalent corresponding to a prime Mersenne number (Mersenne prime $M_p = 2^p - 1$; $(p$: prime) (see: [46, OEIS: `A001348`]), it can be seen from the Table 3 that the decimal equivalent of the beta-based words with a prime value of $n-k$ correspond to a Mersenne prime M_p.

Remark 2.8 Recall that a Toeplitz matrix T is a matrix which has constant values along negative-sloping diagonals and its entries are given by

$$T_{ij} = \begin{cases} c_{i-j+1} & \text{if } i \geq j \\ r_{j-i+1} & \text{otherwise} \end{cases}$$

in which the entry r_1 must be the same as c_1, and this type matrices appear in applications related to approximation theory, signal processing, statistics and time series (*cf.* [53]). In this context, if we consider

$$c_v = 2^{v-1} - 1 \text{ and } r_v = 0$$

for $v \in \mathbb{N}$, the matrix obtained by considering the first $n+1$ rows and first $n+1$ columns in the table to be created with the decimal equivalents of the beta-based words $\mathfrak{B}_w(0;k,n)$ forms an $(n+1)\times(n+1)$ Toeplitz matrix whose entries are defined as follows:

$$T_{ij} = \begin{cases} 2^{i-j} - 1 & \text{if } i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

That's why, the decimal equivalents of the beta-based words are also related to Toeplitz matrices with Mersenne numbers. For example, for $n = k = 5$, the decimal equivalents of the beta-based words $\mathfrak{B}_w(0;k,n)$ generates the following Toeplitz matrix:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 7 & 3 & 1 & 0 & 0 & 0 \\ 15 & 7 & 3 & 1 & 0 & 0 \\ 31 & 15 & 7 & 3 & 1 & 0 \end{bmatrix}.$$

Remark 2.9 Observe also that the values in the Table 4 along negative-sloping diagonals are either Mersenne numbers or the multiplication of Mersenne numbers with a power of 2.

Table 1 Some of the beta-based words $\mathfrak{B}_w(x;k,n)$ in the special case when $x=0$ with $n\in\{0,1,\ldots,8\}$ and $k\in\{0,1,\ldots,6\}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
$n=1$	1	0	ϵ	ϵ	ϵ	ϵ	ϵ
$n=2$	11	01	00	ϵ	ϵ	ϵ	ϵ
$n=3$	111	011	001	000	ϵ	ϵ	ϵ
$n=4$	1111	0111	0011	0001	0000	ϵ	ϵ
$n=5$	11111	01111	00111	00011	00001	00000	ϵ
$n=6$	111111	011111	001111	000111	000011	000001	000000
$n=7$	1111111	0111111	0011111	0001111	0000111	0000011	0000001
$n=8$	11111111	01111111	00111111	00011111	00001111	00000111	00000011

Table 2 Some of the beta-based words $\mathfrak{B}_w(x;k,n)$ in the special case when $x = 1$ with $n \in \{0, 1, \ldots, 8\}$ and $k \in \{0, 1, \ldots, 6\}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
$n=1$	0	1	ϵ	ϵ	ϵ	ϵ	ϵ
$n=2$	00	10	11	ϵ	ϵ	ϵ	ϵ
$n=3$	000	100	110	111	ϵ	ϵ	ϵ
$n=4$	0000	1000	1100	1110	1111	ϵ	ϵ
$n=5$	00000	10000	11000	11100	11110	11111	ϵ
$n=6$	000000	100000	110000	111000	111100	111110	111111
$n=7$	0000000	1000000	1100000	1110000	1111000	1111100	1111110
$n=8$	00000000	10000000	11000000	11100000	11110000	11111000	11111100

Table 3 Decimal equivalents of the beta-based words $\mathfrak{B}_w(x;k,n)$ obtained when $x=0, n\in\{0,1,\ldots,8\}$ and $k\in\{0,1,\ldots,6\}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	0	0	0	0	0	0	0
$n=1$	1	0	0	0	0	0	0
$n=2$	3	1	0	0	0	0	0
$n=3$	7	3	1	0	0	0	0
$n=4$	15	7	3	1	0	0	0
$n=5$	31	15	7	3	1	0	0
$n=6$	63	31	15	7	3	1	0
$n=7$	127	63	31	15	7	3	1
$n=8$	255	127	63	31	15	7	3

Table 4 Decimal equivalents of the beta-based words $\mathfrak{B}_w(x;k,n)$ obtained when $x=1, n\in\{0,1,\ldots,8\}$ and $k\in\{0,1,\ldots,6\}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	0	0	0	0	0	0	0
$n=1$	0	1	0	0	0	0	0
$n=2$	0	2	3	0	0	0	0
$n=3$	0	4	6	7	0	0	0
$n=4$	0	8	12	14	15	0	0
$n=5$	0	16	24	28	30	31	0
$n=6$	0	32	48	56	60	62	63
$n=7$	0	64	96	112	120	124	126
$n=8$	0	128	192	224	240	248	252

Thus, the decimal equivalents of the beta-based words $\mathfrak{B}_w(x;k,n)$ are also related to both the Mersenne numbers and their multiplication with a power of 2. Moreover, the decimal equivalents of the beta-based words $\mathfrak{B}_w(1;k,n)$ form a special $(n+1)\times(n+1)$ matrix whose entries are defined as follows:

$$A_{ij}=\begin{cases}2^{i-j}\left(2^{j-1}-1\right) & \text{if } i\geq j\\ 0 & \text{otherwise.}\end{cases}$$

For example, for $n=k=6$, the decimal equivalents of the beta-based words $\mathfrak{B}_w(1;k,n)$ generates the following special matrix:

$$A=\begin{bmatrix}0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&2&3&0&0&0&0\\0&4&6&7&0&0&0\\0&8&12&14&15&0&0\\0&16&24&28&30&31&0\\0&32&48&56&60&62&63\end{bmatrix}.$$

Now, by representing each successive letter of the beta-based words as a square block with 1s colored black and 0s colored white and stacking square block representations of each beta-based words, we present some plots designed by black-white superimposed square blocks emerging from the first few beta-based words (see Fig. 1). By this way, we present

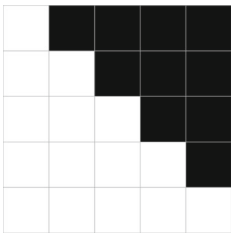
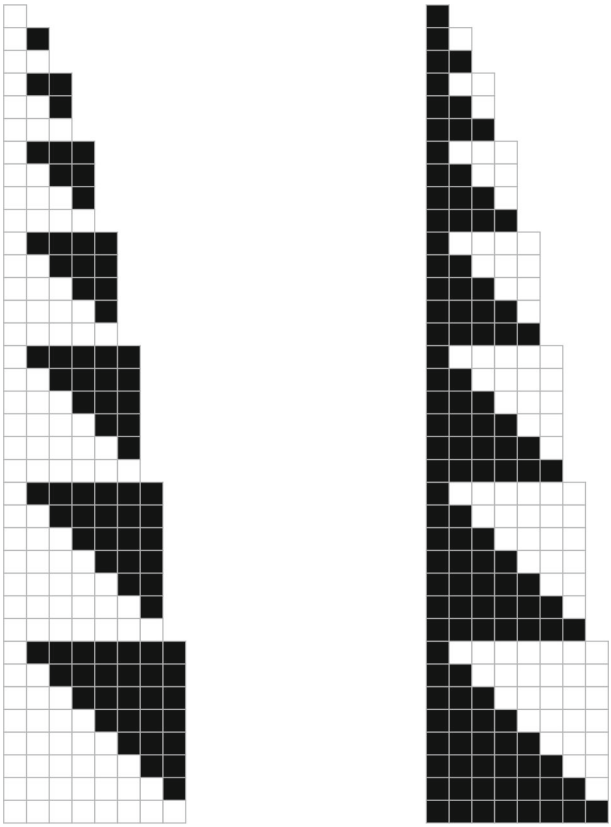


Fig. 1 Black-white superimposed square blocks corresponding to the beta-based word $\mathfrak{B}_w\,(0;k,5)$ when $k \in \{1,2,3,4,5\}$



(a) **(b)**

Fig. 2 Illustrations of the first few beta-based words $\mathfrak{B}_w\,(0;k,5)$ by their black-white superimposed square blocks representations when $n = 1,2,\ldots,8$ and $k = 1,2,\ldots,n$ with the cases **a** $x = 0$ and **b** $x = 1$

two visual illustrations of the beta-based words by stacking strips of black and white squares (See Fig. 2).

Remark 2.10 Observe that Fig. 2a and b are logical complement of each other. Moreover, in Fig. 2a, the number of black squares in the large square obtained for each word of length n forms the sequence $\{0, 1, 3, 6, 10, 15, 21, 28, \ldots\}$ which is given by

$$T(n) = \binom{n}{2} = \frac{n(n-1)}{2}; \quad (n \in \mathbb{N})$$

called Triangular numbers whose details can be found in [47, OEIS: A000217]. On the other hand, in Fig. 2b, the number of black squares in the large square obtained for each word of length n forms the sequence $\{1, 3, 6, 10, 15, 21, 28, 36, \dots\}$ which is corresponding to the sequence $\{T(n+1)\}_{n=1}^{\infty}$ of the Triangular numbers.

Open Question 1: *Are the given illustrations in Figure 2 related to the well-known special sequences of numbers such as the Fibonacci numbers, the Lucas numbers and the Catalan numbers?*

By using Theorem 2.3, we also present Tables 5 and 6 that involves the slopes of the beta-based words $\mathfrak{B}_w(x; k, n)$ for some special cases.

Table 5 Slopes of the beta-based words $\mathfrak{B}_w(x; k, n)$ obtained when $x = 0$ and $n, k \in \{0, 1, \dots, 10\}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$n=1$	0									
$n=2$	$\frac{1}{2}$	0								
$n=3$	$\frac{2}{3}$	$\frac{1}{3}$	0							
$n=4$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0						
$n=5$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0					
$n=6$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0				
$n=7$	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0			
$n=8$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0		
$n=9$	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	0	
$n=10$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{7}{10}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	0

Table 6 Slopes of the beta-based words $\mathfrak{B}_w(x; k, n)$ obtained when $x = 1$ and $n, k \in \{0, 1, \dots, 10\}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$n=1$	1									
$n=2$	$\frac{1}{2}$	1								
$n=3$	$\frac{1}{3}$	$\frac{2}{3}$	1							
$n=4$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1						
$n=5$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1					
$n=6$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1				
$n=7$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	1			
$n=8$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1		
$n=9$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	
$n=10$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	1

Open Question 2: Is there any relation between the Farey sequences and the slopes of the beta-based words? If they have any relationship, in which circumstances do the slopes of the beta-based words form a Farey sequence among themselves?

3 Further results and combinatorial sums arising from the beta polynomials

In this section, we define generalizations of harmonic numbers. By using the beta polynomials and the Bernstein polynomials, we also derive some results and combinatorial sums related to the generalized harmonic numbers.

By the following definition, we first introduce a generalization $\eta_n(a,b)$ of harmonic numbers with real parameters a and b.

Definition 3.1 Let a and b be real parameters. Then, the numbers $\eta_n(a,b)$ are defined by

$$\eta_n(a,b)=\sum_{j=1}^{n}\binom{n}{j}\frac{b^{j-1}}{j+a+1}. \tag{3.1}$$

Some special values of the numbers $\eta_n(a,b)$ are given as follows:

$$\eta_n(-1,-1)=\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{j}=H_n, \tag{3.2}$$

and

$$\eta_n(-1,b)=\frac{1}{b}\sum_{j=1}^{n}\binom{n}{j}\frac{b^j}{j}=\frac{1}{b}\left(-H_n+\sum_{j=1}^{n}\frac{(b+1)^j}{j}\right). \tag{3.3}$$

where H_n denotes the well-known harmonic numbers defined as the sum of the reciprocals of the first n natural numbers.

It is well-known that the numbers H_n arise from truncation of the harmonic series and can be expressed analytically in terms of the Euler–Mascheroni constant and the digamma function. For details related to the harmonic numbers, see [17] and [49], and for details on the achievement of the right-hand sides of the Eqs. (3.2) and (3.2), see [5, p. 126, Eq. (9.27)] and [6].

Next, we set the numbers $c_{n,w}^{(k)}$ by the following finite sum:

$$c_{n,w}^{(k)}:=\sum_{j=1}^{n}(-1)^{j-1}\frac{\binom{n}{j}}{(j+w)^k} \tag{3.4}$$

which, when $w=0$, reduces to the Roman harmonic numbers:

$$c_{n,0}^{(k)}=c_n^{(k)}=\sum_{j=1}^{n}(-1)^{j-1}\frac{\binom{n}{j}}{j^k} \tag{3.5}$$

which was raised by Knuth (*cf.* [14, 17, 26, 33, 36]).

It follows from (3.1) and (3.4) that

$$\eta_n(a,-1)=c_{n,a+1}^{(1)}. \tag{3.6}$$

Remark 3.2 Due to the reduction of the numbers $c_{n,w}^{(k)}$ to the Roman harmonic numbers, the numbers $c_{n,w}^{(k)}$ can be said to be a generalization of the Roman harmonic numbers.

Remark 3.3 In [16], Kim et al. also handled the sums in the following form:

$$\sum_{j=0}^{n}(-1)^k\frac{\binom{n}{j}}{(j+x+1)^m},$$

and for the special cases when $m=2,3,4,5$, Kim et al. [16] showed that the above sums can be expressed in terms of the generalized harmonic functions and the generalized harmonic numbers (See, for details, [16]).

Multiplying both-sides of (1.1) by $(1-x)^{n-k}$, we get

$$(1-x)^{n-k}\mathfrak{B}_{k,n}(x)=x^k\left(1-x^2\right)^{n-k}$$

which, by combining (1.12) and applying the binomial theorem to the final equation, yields

$$\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}x^{n-j}\mathfrak{B}_{k,n}(x)=\frac{1}{\binom{n}{k}}B_k^n(x^2).$$

Integrating both-sides of this equation, wrt x, from 0 to 1, we have

$$\begin{aligned}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\int_0^1 x^{n-j}\mathfrak{B}_{k,n}(x)\mathrm{d}x&=\frac{1}{\binom{n}{k}}\int_0^1 B_k^n(x^2)\mathrm{d}x\\&=\int_0^1 x^{2k}\left(1-x^2\right)^{n-k}\mathrm{d}x.\end{aligned}$$

By making change of variable in the right-hand side of the above equation and combining the final equation with (1.5) and (1.6), we get

$$\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\int_0^1 x^{n-j}\mathfrak{B}_{k,n}(x)\mathrm{d}x=\frac{1}{2}B\left(k+\frac{1}{2},n-k+1\right)\tag{3.7}$$

$$=\frac{1}{2}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(n-k+1\right)}{\Gamma\left(n+\frac{3}{2}\right)}.\tag{3.8}$$

On the other hand, by (1.1), we have

$$\int_0^1 x^{n-j}\mathfrak{B}_{k,n}(x)\mathrm{d}x=\int_0^1 x^{n-j+k}\left(1+x\right)^{n-k}\mathrm{d}x$$

which, by applying the binomial theorem, yields

$$\int_0^1 x^{n-j}\mathfrak{B}_{k,n}(x)\mathrm{d}x=\sum_{v=0}^{n-k}\frac{\binom{n-k}{v}}{2n-j-v+1}.\tag{3.9}$$

By using (3.1) in the right-hand side of the above equation, we arrive at the following theorem:

Theorem 3.4 *Let $n, k, j \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\int_0^1 x^{n-j} \mathfrak{B}_{k,n}(x) \mathrm{d}x = \frac{1}{2n-j+1} - \eta_{n-k}(-2n+j-2,1). \tag{3.10}$$

Blending (3.9) with the left-hand side of (3.8), and then by using (1.8), we also achieve the following theorem:

Theorem 3.5 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{j=0}^{n-k}\sum_{v=0}^{n-k}(-1)^{n-k-j}\frac{\binom{n-k}{j}\binom{n-k}{v}}{2n-j-v+1} = \frac{(n-k)!\Gamma\left(k+\frac{1}{2}\right)}{2\Gamma\left(n+\frac{3}{2}\right)}. \tag{3.11}$$

Combining (3.11) with (3.1), we also get the following result:

Corollary 3.6 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{n-k}\binom{n-k}{v}\eta_{n-k}(-2n+v-2,-1) - \eta_{n-k}(-2n-2,1)\\ &= \frac{(-1)^{n-k}(n-k)!\Gamma\left(k+\frac{1}{2}\right)}{2\Gamma\left(n+\frac{3}{2}\right)} - \frac{1}{2n+1}.\end{aligned} \tag{3.12}$$

From [43], it is known that

$$\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\frac{1}{2n-2j+1} = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma(n-k+1)}{2\Gamma\left(n+\frac{3}{2}\right)} \tag{3.13}$$

for $n, k \in \mathbb{N}_0$ with $n \geq k$ (*cf.* [43, p. 180, Theorem 5.4]).

By combining (1.8) and (3.1) with (3.13), we obtain a formula for the numbers $\eta_n(a,b)$ as in the following theorem:

Theorem 3.7 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\eta_{n-k}\left(-n-\frac{3}{2},-1\right) = \frac{(-1)^{n-k}(n-k)!\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} - \frac{2}{2n+1}. \tag{3.14}$$

By using (1.8) in (3.13), and combining the final equation with (3.11), we achieve the following corollary:

Corollary 3.8 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{j=0}^{n-k}\sum_{v=0}^{n-k}(-1)^{n-k-j}\frac{\binom{n-k}{j}\binom{n-k}{v}}{2n-j-v+1} = \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\frac{1}{2n-2j+1}. \tag{3.15}$$

By combining (3.15) with (3.1), we achieve the following result:

Corollary 3.9 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{v=0}^{n-k}\binom{n-k}{v}\eta_{n-k}(-2n+v-2,-1) = \frac{1}{2}\eta_{n-k}\left(-n-\frac{3}{2},-1\right) + \eta_{n-k}(-2n-2,1). \tag{3.16}$$

Combining (3.11) with (1.9) and (1.10), we get the following result:

Corollary 3.10 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{j=0}^{n-k}\sum_{v=0}^{n-k}(-1)^{n-k-j}\frac{\binom{n-k}{j}\binom{n-k}{v}}{2n-j-v+1}=\frac{4^{n-k}\,(n-k)!\,(2k)!n!}{k!\,(2n+1)!}. \tag{3.17}$$

By combining (3.17) with (3.1), we also achieve the following result:

Corollary 3.11 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{v=0}^{n-k}\binom{n-k}{v}\eta_{n-k}\left(-2n+v-2,-1\right)-\eta_{n-k}\left(-2n-2,1\right) \tag{3.18}$$

$$=\frac{(-4)^{n-k}\,(n-k)!\,(2k)!n!}{k!\,(2n+1)!}-\frac{1}{2n+1}. \tag{3.19}$$

Replacing n by $2n$ and k by n in (3.17), we also have the following result:

Corollary 3.12 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{v=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}\binom{n}{v}}{4n-j-v+1}=\frac{2^{2n}\left((2n)!\right)^2}{(4n+1)!}. \tag{3.20}$$

By combining (3.20) with (3.1), we also achieve the following result:

Corollary 3.13 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{n}\binom{n}{v}\eta_n\left(-4n+v-2,-1\right)-\eta_n\left(-4n-2,1\right) \tag{3.21}$$

$$=\frac{(-1)^n\,2^{2n}\left((2n)!\right)^2-(4n)!}{(4n+1)!}.$$

From [43], it is also known that

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{4n+1-j}=\frac{2^{2n}\left((2n)!\right)^2}{(4n+1)!} \tag{3.22}$$

for $n \in \mathbb{N}_0$ (*cf.* [43, p. 181, Corollary 5.2]).

By combining (3.1) with (3.22), we obtain another formula for the numbers $\eta_n(a, b)$ as in the following theorem:

Theorem 3.14 *Let $n \in \mathbb{N}_0$. Then we have*

$$\eta_n\left(-4n-2,-1\right)=\frac{(-1)^n\,2^{2n}\left((2n)!\right)^2-(4n)!}{(4n+1)!}. \tag{3.23}$$

Combining (3.20) with (3.22) gives the following result:

Corollary 3.15 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{v=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}\binom{n}{v}}{4n-j-v+1}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{1}{4n+1-j}. \tag{3.24}$$

By combining (3.24) with (3.1), we achieve the following result:

Corollary 3.16 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{n} \binom{n}{v} \eta_n (-4n+v-2,-1) = \eta_n (-4n-2,-1) + \eta_n (-4n-2,1). \tag{3.25}$$

Replacing λ by $-x$ in (1.16) and combining the final equation with (1.1), we get the following result that gives a relation between the polynomials $\alpha_k(\lambda; n)$ and the beta polynomials $\mathfrak{B}_{k,n}(x)$:

Theorem 3.17 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\alpha_k (-x; n) = (-1)^{n-k} \frac{x^k (n)_k \mathfrak{B}_{k,n}(x)}{(k+1)!}. \tag{3.26}$$

Integrating both-sides of (3.26), wrt x, from 0 to 1, we have

$$\int_0^1 \alpha_k (-x; n)\, \mathrm{d}x = (-1)^{n-k} \frac{(n)_k}{(k+1)!} \int_0^1 x^k \mathfrak{B}_{k,n}(x) \mathrm{d}x. \tag{3.27}$$

On the other hand, by (1.1), we have

$$\int_0^1 x^k \mathfrak{B}_{k,n}(x) \mathrm{d}x = \int_0^1 x^{2k} (1+x)^{n-k}\, \mathrm{d}x$$

which, by applying the binomial theorem, yields

$$\int_0^1 x^k \mathfrak{B}_{k,n}(x) \mathrm{d}x = \sum_{v=0}^{n-k} \frac{\binom{n-k}{v}}{2k+v+1}. \tag{3.28}$$

By using (3.1) in the right-hand side of the above equation, we get

$$\int_0^1 x^k \mathfrak{B}_{k,n}(x) \mathrm{d}x = \frac{1}{2k+1} + \eta_{n-k} (2k, 1). \tag{3.29}$$

Substituting (3.29) into the right-hand side of (3.27), we achieve the following theorem:

Theorem 3.18 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\int_0^1 \alpha_k (-x; n)\, \mathrm{d}x = (-1)^{n-k} \frac{(n)_k}{(k+1)!} \left(\frac{1}{2k+1} + \eta_{n-k}(2k, 1) \right). \tag{3.30}$$

Remark 3.19 For other integral formulas for the polynomials $\alpha_k(x; n)$, the interested reader may refer to [21].

By the motivation of (3.20), we define the numbers Υ_n by the following explicit formula:

$$\Upsilon_n := \frac{2^{2n} ((2n)!)^2}{(4n+1)!} \tag{3.31}$$

whose computation formulas, due to the combination of (3.20) and (3.22) with (3.1), are given by the following combinatorial sums:

$$\Upsilon_n = \sum_{j=0}^{n} \sum_{v=0}^{n} (-1)^{n-j} \frac{\binom{n}{j}\binom{n}{v}}{4n-j-v+1}. \tag{3.32}$$

and

$$\Upsilon_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{1}{4n+1-j} \tag{3.33}$$

$$= (-1)^n \left(\frac{1}{4n+1} + \eta_n(-4n-2,-1) \right). \tag{3.34}$$

We also give a relation among the numbers Υ_n, the numbers $c_{n,w}^{(k)}$ and the numbers $\eta_n(a,b)$, by the following theorem:

Theorem 3.20 *Let $n \in \mathbb{N}_0$. Then we have*

$$\Upsilon_n = (-1)^n \left(\sum_{v=0}^{n} \binom{n}{v} c_{n,v-4n-1}^{(1)} + \sum_{v=0}^{n} \frac{\binom{n}{v}}{4n-v+1} \right), \tag{3.35}$$

and equivalently

$$\Upsilon_n = (-1)^n \left(\sum_{v=0}^{n} \binom{n}{v} c_{n,v-4n-1}^{(1)} + \frac{1}{4n+1} - \eta_n(-4n-2,1) \right). \tag{3.36}$$

Proof By rearranging (3.32), we have

$$\Upsilon_n = \sum_{v=0}^{n} \sum_{j=1}^{n} (-1)^{n-j} \frac{\binom{n}{j}\binom{n}{v}}{4n-j-v+1} + (-1)^n \sum_{v=0}^{n} \frac{\binom{n}{v}}{4n-v+1}. \tag{3.37}$$

Using (3.4) and (3.1) in the above equation, we achieve the assertion of Theorem 3.20. □

By using the explicit formula (3.31) for the numbers Υ_n in (3.36), we also get the following result:

Corollary 3.21 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{n} \binom{n}{v} c_{n,v-4n-1}^{(1)} - \eta_n(-4n-2,1) = (-1)^n \frac{2^{2n}\,((2n)!)^2}{(4n+1)!} - \frac{1}{4n+1}. \tag{3.38}$$

Remark 3.22 Using (3.6) in the above equation, the identity in (3.21) can also be obtained.

4 Conclusions and discussions

Hereby, we introduced a new class of combinatorial objects, termed beta-based words, inspired by the beta polynomials. Moreover, we developed a generation algorithm; provided computational implementations in the Wolfram language, and presented numerical tables of the resulting words along with their decimal equivalents. In addition, in this paper, several structural properties of the beta-based words—such as complement, symmetry, length, height, and slope—have been examined in detail. We also explored connections between the decimal equivalents of these words and Mersenne numbers, presented visual illustrations using black-and-white square block plots, and derived related combinatorial sums involving the beta polynomials, the Bernstein polynomials, and a generalized form of harmonic numbers.

In conjunction with these findings of this paper, we also raised a few open questions related to the potential relations of the beta-based words. In addition to the open questions raised in this paper, we also raise other open questions that could offer the reader new areas of study in algebraic manner:

Could the concept of "beta-based words" be of any help to solve the word problem in monoids using Gröbner-Shirshov bases?

Is there any formula that associate the "beta-based words" with special graphs and monoids?

For details on the Gröbner-Shirshov bases, the reader may refer [2, 15, 18]. Additionally, for details on special graphs and monoids, the interested reader may refer [8, 10, 11].

In conclusion, the scope of this paper lies at the intersection of combinatory, special polynomials, and symbolic computation, and has a high potential to be of interest to researchers working currently active in these areas due to its relevance to modern technological applications. Therefore, the findings of this paper can be effectively applied across a wide range of modeling applications when blended in conjunction with appropriate techniques.

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Conflict of interest The authors declare no conflict of interest.

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