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# ON REFLECTIONS AND ROTATIONS IN MINKOWSKI 3-SPACE

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**Abstract.** In this paper, we investigate the reflections in Minkowski 3-space by three different appoach. Firstly, we define Lorentzian reflections with Lorentzian inner product. Then, we examine Lorentzian reflections in view of Lorentzian Householder matrices. Finally, we use pure split quaternions to derive Lorentzian reflections. For each case, we find the matrix representation of Lorentzian reflections and characterize the plane of reflection by using this matrix representation. Moreover, we prove that any Lorentzian orthogonal transformation can be represented by the composition of at most six reflections.

*MSC*: 15B10, 15A16, 53B30.

*Keywords*: Minkowski Space, Reflections, Rotations.

## 1. Introduction

In the Euclidean space, a reflection is an isometry with a hyperplane as a set of fixed points; this set is called the axis (in dimension 2) or plane (in dimension 3) of reflection. The image of a figure by a reflection is its mirror image in the axis or plane of reflection. A reflection along a subspace in $\mathbb{R}^n$ maintains the perpendicular distance from this subspace and its orthogonal complement. And orthogonal complement of this subspace separates the initial and reflected vectors. Thus the components of the initial vector orthogonal to the subspace along which the reflection occurs remain unchanged.

A reflection in $\mathbb{R}^n$ can be represented by an $n \times n$ symmetric orthogonal matrix with determinant -1 and eigenvalues (1, 1, 1, ... 1, -1). The product of two such matrices is a special orthogonal matrix which represents a rotation. Every rotation is the result of reflecting in an even number of reflections in hyperplanes through the origin. Thus, the reflections generate orthogonal group and this result is known as the Cartan–Dieudonné theorem. In the study [3], answers of the following two frequently asked questions are handled;

Question 1. *"In how many ways can an orthogonal transformation in an n-dimensional Euclidean space be decomposed?"*

Question 2. *"What are the simple reflections that determine a given orthogonal transformation in* $\mathbb{R}^n$*?"* Or, in particular *"What Householder matrices determine a given orthogonal matrix in* $\mathbb{R}^n$*?"*
Moreover, a constructive proof of the Cartan Theorem is provided with the relationship of Pythagorean numbers by using Clifford algebras. Then, simple reflections or rotations of canonical vectors are used to generate all Pythagorean vectors and a method is developed to generate generalized Pythagorean numbers in [4]. On the other hand, an algorithmic proof of the Cartan-Dieudonné Theorem is given on generalized real scalar product spaces with arbitrary signature In the study [9]. Then, Clifford algebras is used to compute the factorization of a given orthogonal transformation as a product of reflections with respect to hyperplanes in [9].

In the study [6], rotation matrices in Minkowski 3-space are generated with unit timelike split quaternions. Then, the eigenvalues and eigenvector of a Lorentzian rotation matrix are examined in term of the coefficients of the corresponding quaternion in [8]. Moreover, the characterizations of eigenvalues (complex or real) of a Lorentzian rotation matrix is also given according to only first component of the corresponding unit timelike quaternion in [8].

This paper is concerned with the reflections and rotations in Minkowski 3-space. Firstly, we investigate the Lorentzian reflections by three different approach. Firstly, we examine Lorentzian reflections with Lorentzian inner product. Then, we express Lorentzian reflections in view of Lorentzian Householder matrices. Finally, we use pure split quaternions to derive Lorentzian reflections. For each case, we find the matrix representation of Lorentzian reflections and characterize the plane of reflection by using this matrix representation. Moreover, we prove that any Lorentzian orthogonal transformation can be represented by the composition of at most six reflections.

## 2. Preliminaries

In this part, a brief summary of Minkowski 3-space and the split quaternions is presented to provide the necessary background.

The Minkowski 3-space is a Euclidean 3-space, $\mathbb{E}^3$, equipped with Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}} = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}_1^3$. The norm of the vector $\mathbf{u} \in \mathbb{E}_1^3$ is defined by

$$\|u\| = \sqrt{|\langle u, u \rangle_{\mathbb{L}}|}.$$

Lorentzian inner product characterizes the elements $\mathbf{u} = (u_1, u_2, u_3)$ of $\mathbb{E}_1^3$ as follows;

if $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} > 0$ or $\mathbf{u} = 0$ then $\mathbf{u}$ is called *spacelike*,

if $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} < 0$ then $\mathbf{u}$ is called *timelike*,

if $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} = 0$ and $\mathbf{u} \neq 0$ then $\mathbf{u}$ is called *lightlike* or *null*.

**Proposition 1** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two nonnull vectors in* $\mathbb{E}_1^3$ *such that* $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$. *Then the followings are satisfied;*
***i.*** $(\mathbf{u} - \mathbf{v}) \perp_{\mathbb{L}} (\mathbf{u} + \mathbf{v})$,
***ii.*** *At least one of the vectors* $(\mathbf{u} - \mathbf{v})$ *and* $(\mathbf{u} + v)$ *is a nonnull vector.*

**Proof:** Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two nonnull vectors in $\mathbb{E}_1^3$ such that $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$.
**i.**

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{\mathbb{L}} = \varepsilon_{\mathbf{u}} + \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}} - \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}} - \varepsilon_{\mathbf{v}} = 0.$$

This means that $(\mathbf{u} - \mathbf{v}) \perp_L (\mathbf{u} + \mathbf{v})$.
**ii.** We know that

$$\varepsilon_{\mathbf{u}+\mathbf{v}} = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle_{\mathbb{L}} = 2(\varepsilon_{\mathbf{u}} + \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}}),$$
$$\varepsilon_{\mathbf{u}-\mathbf{v}} = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{\mathbb{L}} = 2(\varepsilon_{\mathbf{u}} - \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}}).$$

Using these relations, we have

$$\varepsilon_{\mathbf{u}+\mathbf{v}} + \varepsilon_{\mathbf{u}-\mathbf{v}} = 4\varepsilon_{\mathbf{u}} = 4\varepsilon_{\mathbf{v}} \neq 0.$$

Thus, we get $(\mathbf{u} - \mathbf{v})$ or $(\mathbf{u} + \mathbf{v})$ is a nonnull vector. ■

For any $R \in M_{3\times 3}(\mathbb{R})$, if $\langle R\mathbf{u}, R\mathbf{u} \rangle_{\mathbb{L}} = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}}$ for all vectors $\mathbf{u} \in \mathbb{E}_1^3$, then $R$ is called a semi orthogonal matrix. That is, semi orthogonal matrices preserve the length of vectors in the Minkowski 3-space and columns (or rows) of the a semi orthogonal matrix form an orthonormal base of $\mathbb{E}_1^3$. Moreover, $R$ is a semi orthogonal matrix if and only if $I^* R^T I^* = R^{-1}$ where $R^T$ is transpose of the matrix $R$ and

$$I^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, $R$ is a semi orthogonal matrix if and only if $R^T I^* R = I^*$. Taking the determinant of both sides of this equation gives $\det(R) = \pm 1$. If $\det R = 1$, then $R$ is called Lorentzian rotation matrix in $\mathbb{E}_1^3$. The set of the Lorentzian rotation matrices of $\mathbb{E}_1^3$ can be expressed as follows:

$$SO(1,2) = \{R \in M_{3\times 3}(\mathbb{R}) : R^T I^* R = I^* \text{ and } \det R = 1\}.$$

On the other hand, for any $A \in M_{3\times 3}(\mathbb{R})$, if $A$ satisfies the equality $I^*A^TI^* = A$, then $A$ is called a semi symmetric matrix in Minkowski 3-space. In general, a semi symmetric matrix $A$ can be in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Notice that, the above matrix $A$ satisfies the equality $I^*A^TI^* = A$. If $\det A = -1$, then $A$ is called Lorentzian reflection matrix in $\mathbb{E}_1^3$.

The set of split quaternions, which was introduced by James Cockle in 1849 [2], can be represented as

$$\widehat{\mathbb{H}} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ \ q_0, q_1, q_2, q_3 \in \mathbb{R}\}.$$

Here the imaginary units satisfy the following relations:

$$i^2 = -1,\ j^2 = k^2 = ijk = 1,$$
$$ij = -ji = k,\ jk = -kj = -i,\ ki = -ik = j.$$

Unlike quaternion algebra, the set of split quaternions contains zero divisors, nilpotent elements and nontrivial idempotents [7]. For any $q = q_0+q_1i+q_2j+q_3k \in \widehat{\mathbb{H}}$, we define scalar part of $q$ as $S_q = q_0$, vector part of $q$ as $V_q = q_1i + q_2j + q_3k$ and conjugate of $q$ as $\overline{q} = S_q - V_q = q_0 - q_1i - q_2j - q_3k$. If $S_q = 0$, then $q$ is called pure split quaternion. And any pure split quaternion $q = q_1i + q_2j + q_3k$ can be represented as $\mathbf{q} = (q_1, q_2, q_3)$. Furthermore, any pure split quaternion can be considered as a vector in $\mathbb{E}_1^3$. That is the set of pure split quaternions is identified with Minkowski 3-space. The sum and product of two split quaternions $p = p_0 + p_1i + p_2j + p_3k$ and $q = q_0 + q_1i + q_2j + q_3k$ are defined as

$$\begin{aligned} p + q &= S_p + S_q + V_p + V_q \\ &= (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3), \\ pq &= S_pS_q + \langle V_p, V_q\rangle_{\mathbb{L}} + S_pV_q + S_qV_p + V_p \times_{\mathbb{L}} V_q \\ &= (p_0q_0 - p_1q_1 + p_2q_2 + p_3q_3) + (p_1q_0 + p_0q_1 - p_2q_3 + p_3q_2)i \\ &+ (p_0q_2 + p_2q_0 - p_1q_3 + p_3q_1)j + (p_0q_3 + p_3q_0 - p_2q_1 + p_1q_2)k, \end{aligned}$$

respectively. Here $\times_{\mathbb{L}}$ denotes the Lorentzian vector product and defined as

$$V_p \times_{\mathbb{L}} V_q = \begin{vmatrix} -i & j & k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix},$$

respectively. The term $I_q = q\overline{q} = \overline{q}q = q_0^2 + q_1^2 - q_2^2 - q_3^2$ characterizes any split quaternion $q$. The split quaternion $q$ is called spacelike, timelike or null, if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively. And the norm of $q$ is defined as

$$N_q = \sqrt{|I_q|} = \sqrt{|q_0^2 + q_1^2 - q_2^2 - q_3^2|}.$$

If $N_q = 1$, then $q$ is called unit split quaternion. If $I_q \neq 0$, then $q^{-1} = \dfrac{\overline{q}}{I_q}$ [7], [8].

## 3. Lorentzian Reflections

In this part, we investigate the reflections in Minkowski 3-space by three different approach. Firstly, we define Lorentzian reflections with Lorentzian inner product. Then, we examine Lorentzian reflections in view of Lorentzian Householder matrices. Finally, we use pure split quaternions to derive Lorentzian reflections.

### 3.1. Reflections by Lorentzian inner product

Let us define a simple reflection in Minkowski 3-space by using Lorentzian inner product. Consider the orthogonal complement (plane) of the vector $\mathbf{x}$, defined by

$$\mathbf{x}^{\perp} = \left\{\mathbf{y} \in \mathbb{E}_1^3 : \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{L}} = 0\right\}.$$

For any nonzero and collinear vectors $\mathbf{x}$ and $y$ in $\mathbb{E}_1^3$. The projection of a vector $y$ onto $\mathbf{x}^{\perp}$ is $y - r\mathbf{x}$ where $r \in \mathbb{R}$ is chosen such that $y - r\mathbf{x} \in \mathbf{x}^{\perp}$. That is

$$\langle \mathbf{y} - r\mathbf{x}, \mathbf{x} \rangle_{\mathbb{L}} = 0 \Longrightarrow r = \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{L}}}{\varepsilon_{\mathbf{x}}}$$

where $\mathbf{x}$ is nonnull and $\varepsilon_{\mathbf{x}} = \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{L}}$ . In this way, the reflection of $y$ through the plane $\mathbf{x}^{\perp}$ is called a simple reflection and is given by

$$\varphi_{\mathbf{x}}(\mathbf{y}) = \mathbf{y} - 2r\mathbf{x} = \mathbf{y} - 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{L}}}{\varepsilon_{\mathbf{x}}}\mathbf{x}.$$

It is essential to note that if $y \in \mathbf{x}^{\perp}$, then $\varphi_{\mathbf{x}}(y) = y$. This means $\varphi_{\mathbf{x}}$ acts as identity map on the plane $\mathbf{x}^{\perp}$. Now, let us show that the inverse of a simple reflection is itself, that is for any $y \in \mathbb{E}_1^3$, $\varphi_{\mathbf{x}}(\varphi_{\mathbf{x}}(y)) = y$.

$$\begin{aligned}
\varphi_{\mathbf{x}}(\varphi_{\mathbf{x}}(\mathbf{y})) &= \varphi_{\mathbf{x}}(\mathbf{y} - 2r\mathbf{x}) \\
&= \mathbf{y} - 2r\mathbf{x} - 2\frac{\langle \mathbf{x}, \mathbf{y} - 2r\mathbf{x} \rangle_{\mathbb{L}}}{\varepsilon_{\mathbf{x}}}\mathbf{x} \\
&= \mathbf{y} - 2r\mathbf{x} - 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{L}} - 2r\langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{L}}}{\varepsilon_{\mathbf{x}}}\mathbf{x} \\
&= \mathbf{y} - 2r\mathbf{x} - 2r\mathbf{x} + 4r\mathbf{x} = \mathbf{y}.
\end{aligned}$$

Moreover, $\varphi_{\mathbf{x}}$ is an orthogonal transformation. Since

$$\begin{aligned}\|\varphi_{\mathbf{x}}(\mathbf{y})\|^2 &= \langle \mathbf{y}-2r\mathbf{x}, \mathbf{y}-2r\mathbf{x}\rangle_{\mathbb{L}} \\ &= \langle \mathbf{y},\mathbf{y}\rangle_{\mathbb{L}} - 4r\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{L}} + 4r^2\langle \mathbf{x},\mathbf{x}\rangle_{\mathbb{L}} \\ &= \langle \mathbf{y},\mathbf{y}\rangle_{\mathbb{L}} - \frac{4}{\varepsilon_{\mathbf{x}}}(\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{L}})^2 + \frac{4}{\varepsilon_{\mathbf{x}}}(\langle \mathbf{x},\mathbf{x}\rangle_{\mathbb{L}})^2 \\ &= \langle \mathbf{y},\mathbf{y}\rangle_{\mathbb{L}} = \|\mathbf{y}\|^2.\end{aligned}$$

**Theorem 2** *For any two nonnull vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{E}_1^3$*, such that* $\mathbf{u} \neq \mathbf{v}$ *and* $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$*, there exists a linear map* $\varphi$ *such that* $\varphi(\mathbf{u}) = \mathbf{v}$ *and* $\varphi$ *is a Lorentzian reflection or the compsition of two Lorentzian reflections.*

**Proof:** Let $\mathbf{u}$ and $\mathbf{v}$ be two nonnull different vectors in $\mathbb{E}_1^3$ such that $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$. Consider the linear map $\varphi : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ defined as

$$\varphi(\mathbf{x}) = \begin{cases} \varphi_{\mathbf{u}}(\mathbf{x}) & \text{if } \mathbf{v} = -\mathbf{u} \\ \varphi_{\mathbf{u}-\mathbf{v}}(\mathbf{x}) & \text{if } \mathbf{v} \neq -\mathbf{u} \text{ and } (\mathbf{u}-\mathbf{v}) \text{ is nonnull} \\ \varphi_{\mathbf{v}}\varphi_{\mathbf{u}+\mathbf{v}}(\mathbf{x}) & \text{if } \mathbf{v} \neq -\mathbf{u} \text{ and } (\mathbf{u}-\mathbf{v}) \text{ is null} \end{cases}. \tag{1}$$

If $\mathbf{v} = -\mathbf{u}$, then

$$\varphi(\mathbf{u}) = \varphi_{\mathbf{u}}(\mathbf{u}) = \mathbf{u} - 2\frac{\langle \mathbf{u},\mathbf{u}\rangle_{\mathbb{L}}}{\langle \mathbf{u},\mathbf{u}\rangle_{\mathbb{L}}}\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u} = \mathbf{v}.$$

If $\mathbf{v} \neq -\mathbf{u}$ and $(\mathbf{u}-\mathbf{v})$ is a nonnull vector, then

$$\varphi(\mathbf{u}) = \varphi_{\mathbf{u}-\mathbf{v}}(\mathbf{u}) = \mathbf{u} - 2\frac{\langle \mathbf{u},\mathbf{u}-\mathbf{v}\rangle_{\mathbb{L}}}{\langle \mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle_{\mathbb{L}}}(\mathbf{u}-\mathbf{v})$$

$$= \mathbf{u} - 2\frac{\langle \mathbf{u},\mathbf{u}\rangle_{\mathbb{L}} - \langle \mathbf{u},\mathbf{v}\rangle_{\mathbb{L}}}{2(\langle \mathbf{u},\mathbf{u}\rangle_{\mathbb{L}} - \langle \mathbf{u},\mathbf{v}\rangle_{\mathbb{L}})}(\mathbf{u}-\mathbf{v})$$

$$= \mathbf{u} - (\mathbf{u}-\mathbf{v}) = \mathbf{v}.$$

If $\mathbf{v} \neq -\mathbf{u}$ and $(\mathbf{u}-\mathbf{v})$ is a null vector, then

$$\varphi_{\mathbf{u}+\mathbf{v}}(\mathbf{u}) = \mathbf{u} - 2\frac{\langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle_{\mathbb{L}}}{\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle_{\mathbb{L}}}(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u} - 2\frac{\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}}}{2(\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}})}(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u} - (\mathbf{u}+\mathbf{v}) = -\mathbf{v}.$$

$$\varphi(\mathbf{u}) = \varphi_{\mathbf{v}}(-\mathbf{v}) = -\mathbf{v} - 2\frac{\langle \mathbf{v}, -\mathbf{v}\rangle_{\mathbb{L}}}{\langle \mathbf{v}, \mathbf{v}\rangle_{\mathbb{L}}}\mathbf{v} = \mathbf{v}.$$

By above discussions and construction of the map $\varphi$ is a Lorentzian reflection or composition of two Lorentzian reflection such that $\varphi(\mathbf{u}) = \mathbf{v}$. ■

**Remark 3** *The case* $\mathbf{u} = \mathbf{v}$ *is special. In general, there are infinitely many different linear map that leave* $\mathbf{u}$ *fixed. For this reason, in the remainder of this paper, for the case* $\mathbf{u} = \mathbf{v}$ *we will take* $\varphi$ *as the identity operator.*

**Theorem 4** *The matrix representation of Lorentzian reflection* $\varphi_{\mathbf{x}}$ *through the hyperplane* $x^{\perp}$ *is obtained as follows;*

$$\varphi_{\mathbf{x}} = \frac{1}{\varepsilon_{\mathbf{x}}}\begin{bmatrix} x_1^2+x_2^2+x_3^2 & -2x_1x_2 & -2x_1x_3 \\ 2x_1x_2 & -x_1^2-x_2^2+x_3^2 & -2x_2x_3 \\ 2x_1x_3 & -2x_2x_3 & -x_1^2+x_2^2-x_3^2 \end{bmatrix}$$

*where* $\mathbf{x} = (x_1, x_2, x_3)$.

**Proof:** Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{E}_1^3$ and $\mathbf{x} = (x_1, x_2, x_3)$. Then we have

$$\varphi_{\mathbf{x}}(e_1) = e_1 - 2\frac{\langle \mathbf{x}, e_1\rangle_L}{\varepsilon_{\mathbf{x}}}\mathbf{x} = (1+2\frac{x_1^2}{\varepsilon_{\mathbf{x}}}, 2x_1\frac{x_2}{\varepsilon_{\mathbf{x}}}, 2x_1\frac{x_3}{\varepsilon_{\mathbf{x}}}),$$
$$\varphi_{\mathbf{x}}(e_2) = e_2 - 2\frac{\langle \mathbf{x}, e_2\rangle_L}{\varepsilon_{\mathbf{x}}}\mathbf{x} = (-2x_1\frac{x_2}{\varepsilon_{\mathbf{x}}}, 1-2\frac{x_2^2}{\varepsilon_{\mathbf{x}}}, -2x_2\frac{x_3}{\varepsilon_{\mathbf{x}}}),$$
$$\varphi_{\mathbf{x}}(e_2) = e_3 - 2\frac{\langle \mathbf{x}, e_3\rangle_L}{\varepsilon_{\mathbf{x}}}\mathbf{x} = (-2x_1\frac{x_3}{\varepsilon_{\mathbf{x}}}, -2x_2\frac{x_3}{\varepsilon_{\mathbf{x}}}, 1-2\frac{x_3^2}{\varepsilon_{\mathbf{x}}}).$$

So we get the matrix representation of the Lorentzian reflection as follows:

$$\varphi_{\mathbf{x}} = \frac{1}{\varepsilon_{\mathbf{x}}}\begin{bmatrix} x_1^2+x_2^2+x_3^2 & -2x_1x_2 & -2x_1x_3 \\ 2x_1x_2 & -x_1^2-x_2^2+x_3^2 & -2x_2x_3 \\ 2x_1x_3 & -2x_2x_3 & -x_1^2+x_2^2-x_3^2 \end{bmatrix}.$$

■

Notice that the matrix $\varphi_{\mathbf{x}}$ has determinant $-1$ and satisfies the relation $I^*\varphi_{\mathbf{x}}I^* = \varphi_{\mathbf{x}}^T$. And the Lorentzian reflection $\varphi_{\mathbf{x}}$ through the hyperplane $x^\perp$ has the set of eigenvalues $\{-1, 1, 1\}$ with the corresponding eigenvectors

$$\{(x_1, x_2, x_3), (x_2, x_1, 0), (x_3, 0, x_1)\}.$$

In other words, the plane of reflection can be determined by the eigenvector of the Lorentzian reflection matrix corresponding to the eigenvalue $-1$.

### 3.2. Reflections by Lorentzian Householder Transformation

Let $I$ be $3 \times 3$ identity matrix and $I^* = diag\{-1, 1, 1\}$. The Lorentzian Householder transformation associated with the nonnull vector $\mathbf{x} =(x_1, x_2, x_3) \in \mathbb{E}_1^3$ is defined as

$$H_{\mathbf{x}} = I - \frac{2}{\varepsilon_{\mathbf{x}}}\mathbf{x}\mathbf{x}^t I^*.$$

The Lorentzian Householder transformation associated with the nonnull vector $\mathbf{x}$ has the following matrix representation;

$$H_{\mathbf{x}} = \frac{1}{\varepsilon_{\mathbf{x}}}\begin{bmatrix} x_1^2 + x_2^2 + x_3^2 & -2x_1x_2 & -2x_1x_3 \\ 2x_1x_2 & -x_1^2 - x_2^2 + x_3^2 & -2x_2x_3 \\ 2x_1x_3 & -2x_2x_3 & -x_1^2 + x_2^2 - x_3^2 \end{bmatrix}.$$

It can be easily seen that Lorentzian Householder transformation $H_{\mathbf{x}}$ has the following properties:

- it is semi symmetric: $I^*H_{\mathbf{x}}I^* = H^T$,
- it is semi orthogonal: $I^*H_{\mathbf{x}}^{-1}I^* = H^T$,
- hence it is involutory: $H_{\mathbf{x}}^2 = I$.

Moreover, $H_{\mathbf{x}}$ has the set of eigenvalues $\{-1, 1, 1\}$ with the corresponding eigenvectors $\{(x_1, x_2, x_3),\ (x_2, x_1, 0),\ (x_3, 0, x_1)\}$, respectively. The determinant of $H_{\mathbf{x}}$ is $-1$, since the determinant of a matrix is the product of its eigenvalues. Therefore, Lorentzian Householder transformation $H_{\mathbf{x}}$ represents the Lorentzian reflection where the hyperplane of the reflection is $\mathbf{x}^\perp$. The reflection of a point $\mathbf{u} \in \mathbb{E}_1^3$ about this hyperplane is:

$$H_{\mathbf{x}}(\mathbf{u}) = \mathbf{u} - \frac{2}{\varepsilon_{\mathbf{x}}}\mathbf{x}\mathbf{x}^t I^*\mathbf{u}.$$

**Theorem 5** *For any two nonnull vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{E}_1^3$*, such that* $\mathbf{u} \neq \mathbf{v}$ *and* $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$*, there exists a linear map* $H$ *such that* $H(\mathbf{u}) = \mathbf{v}$ *and* $H$ *is either a Lorentzian Householder transformation or composition of two Lorentzian Householder transformations*.

**Proof:** Let $\mathbf{u}$ and $\mathbf{v}$ be two nonnull different vectors in $\mathbb{E}_1^3$ such that $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$. Consider the linear map $H : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ defined as

$$H(\mathbf{x}) = \begin{cases} H_{\mathbf{u}}(\mathbf{x}) & \text{if } \mathbf{v} = -\mathbf{u} \\ H_{\mathbf{u}-\mathbf{v}}(\mathbf{x}) & \text{if } \mathbf{v} \neq -\mathbf{u} \text{ and } (\mathbf{u}-\mathbf{v}) \text{ is nonnull} \\ H_{\mathbf{v}}(H_{\mathbf{u}+\mathbf{v}}(\mathbf{x})) & \text{if } \mathbf{v} \neq -\mathbf{u} \text{ and } (\mathbf{u}-\mathbf{v}) \text{ is null} \end{cases}. \tag{2}$$

Then If $\mathbf{v} = -\mathbf{u}$, then

$$H(\mathbf{u}) = H_{\mathbf{u}}(\mathbf{u}) = \mathbf{u} - \frac{2}{\varepsilon_{\mathbf{u}}}\mathbf{u}\mathbf{u}^t I^*\mathbf{u} = \mathbf{u} - \frac{2}{\varepsilon_{\mathbf{u}}}\mathbf{u}(\varepsilon_{\mathbf{u}}) = -\mathbf{u} = \mathbf{v}.$$

If $\mathbf{v} \neq -\mathbf{u}$ and $(\mathbf{u}-\mathbf{v})$ is a nonnull vector, then we get

$$H(\mathbf{u}) = H_{\mathbf{u}-\mathbf{v}}(\mathbf{u}) = \mathbf{u} - \frac{2}{\varepsilon_{\mathbf{u}-\mathbf{v}}}(\mathbf{u}-\mathbf{v})(\mathbf{u}-\mathbf{v})^T I^*\mathbf{u}$$

$$= \mathbf{u} - \frac{2\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} - \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}}}{2(\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} - \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}})}(\mathbf{u}-\mathbf{v})$$

$$= \mathbf{u} - (\mathbf{u}-\mathbf{v}) = \mathbf{v}$$

by using the identities $\mathbf{u}^t I^*\mathbf{u} = \mathbf{v}^t I^*\mathbf{v} = \varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$ and $\mathbf{u}^t I^*\mathbf{v} = \mathbf{v}^t I^*\mathbf{u} = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}}$. If $\mathbf{v} \neq -\mathbf{u}$ and $(\mathbf{u}-\mathbf{v})$ is a null vector, then we obtain

$$H_{\mathbf{u}+\mathbf{v}}(\mathbf{u}) = \mathbf{u} - \frac{2}{\varepsilon_{\mathbf{u}+\mathbf{v}}}(\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v})^T I^*\mathbf{u}$$

$$= \mathbf{u} - \frac{2(\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}})}{2(\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{L}})}(\mathbf{u}+\mathbf{v})$$

$$= -\mathbf{u} - (\mathbf{u}+\mathbf{v}) = -\mathbf{v}.$$

$$H(\mathbf{u}) = H_{\mathbf{v}}(-\mathbf{v}) = -\mathbf{v} - \frac{2}{\varepsilon_{\mathbf{v}}}\mathbf{v}\mathbf{v}^T I^*(-\mathbf{v}) = \mathbf{v}.$$

By above discussions and construction of the linear map $H$ is a the Lorentzian Householder transformation or composition of two Lorentzian Householder transformation such that $H(\mathbf{u}) = \mathbf{v}$. ■

### 3.3. Lorentzian Reflections by Pure Split Quaternions

Let us define a simple reflection in Minkowski 3-space by pure split quaternion. The set of pure split quaternions is denoted by

$$\mathrm{Im}\widehat{\mathbb{H}} = \{q = q_1 i + q_2 j + q_3 k \in \widehat{\mathbb{H}}\}.$$

Since $\mathrm{Im}\widehat{\mathbb{H}}$ is identified with Minkowski 3-space, then any $x = x_1 i + x_2 j + x_3 k \in \mathrm{Im}\widehat{\mathbb{H}}$ can be considered as a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{E}_1^3$. Now, consider the linear transformation

$$\begin{aligned} \phi_x : \mathrm{Im}\widehat{\mathbb{H}} &\to \mathrm{Im}\widehat{\mathbb{H}} \\ : u &\to \phi_x(u) = -\frac{1}{x^2} xux \end{aligned}$$

where $x$ is a nonnull pure split quaternion. Let $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ be the standard basis of $\mathbb{E}_1^3$. Then we have

$$\begin{aligned} \phi_x(i) &= \frac{1}{x^2}[(x_1^2 + x_2^2 + x_3^2)i + (2x_1x_2)j + (2x_1x_3)k], \\ \phi_x(j) &= \frac{1}{x^2}[(-2x_1x_2)i + (-x_1^2 - x_2^2 + x_3^2)j + (-2x_2x_3)k)], \\ \phi_x(k) &= \frac{1}{x^2}[(-2x_1x_3)i + (2x_2x_3)j + (-x_1^2 + x_2^2 - x_3^2)k)]. \end{aligned}$$

The matrix representation of the transformation $\phi_{\mathbf{x}}$ is obtained as follows;

$$\phi_{\mathbf{x}} = \frac{1}{x^2}\begin{bmatrix} x_1^2 + x_2^2 + x_3^2 & -2x_1x_2 & -2x_1x_3 \\ 2x_1x_2 & -x_1^2 - x_2^2 + x_3^2 & -2x_2x_3 \\ 2x_1x_3 & -2x_2x_3 & -x_1^2 + x_2^2 - x_3^2 \end{bmatrix}.$$

We can easily see that the linear transformation $\phi_x$ satisfies the relation $\phi_x^T I^* \phi_x = I^*$ and $\det\phi_x = -1$. And $\phi_x$ has the set of eigenvalues $\{-1, 1, 1\}$ with the corresponding eigenvectors $\{(x_1, x_2, x_3), (x_2, x_1, 0), (x_3, 0, x_1)\}$, respectively. Therefore, the linear transformation $\phi_{\mathbf{x}}$ represents the Lorentzian reflection where the hyperplane of the reflection is $\mathbf{x}^{\perp}$.

**Theorem 6** *For any two nonnull pure split quaternions $u$ and $v$, such that $u \neq v$ and $u^2 = v^2$, there exists a linear map $\phi$ such that $\phi(u) = v$ and $\phi$ is either Lorentzian reflection or composition of two Lorentzian reflections.*

**Proof:** Let $u$ and $v$ be two nonnull pure split quaternions such that $u^2 = v^2$. Consider the linear map $\phi : \mathrm{Im}\,\widehat{\mathbb{H}} \to \mathrm{Im}\,\widehat{\mathbb{H}}$ defined as

$$\phi(x) = \begin{cases} \phi_u(x) & \text{if } v = -u \\ \phi_{u-v}(x) & \text{if } v \neq -u \text{ and } (u-v) \text{ is nonnull} \\ \phi_v\phi_{u+v}(\mathbf{x}) & \text{if } v \neq -u \text{ and } (u-v) \text{ is null} \end{cases}. \tag{2}$$

If $v = -u$, then we get

$$\phi(u) = \phi_u(u) = -\frac{1}{u^2}uuu = -u = v.$$

If $v \neq -u$ and $(u - v)$ is a nonnull vector, then we get

$$\phi(u) = \phi_{u-v}(u) = -\frac{1}{(u-v)^2}(u-v)u(u-v)$$

$$= \frac{(2v^2 - uv - vu)}{(2v^2 - uv - vu)}v = v.$$

by using the identities $u^2 = v^2$ and $2v^2 - uv - vu \in \mathbb{R}$. If $\mathbf{v} \neq -\mathbf{u}$ and $(\mathbf{u} - \mathbf{v})$ is a null vector, then we obtain

$$\phi_{u+v}(u) = -\frac{1}{(u+v)^2}(u+v)u(u+v)$$

$$= -\frac{(2v^2 + uv + vu)}{(2v^2 + uv + vu)}v = -v.$$

$$\phi(u) = \phi_v(-v) = -\frac{1}{v^2}v(-v)v = v.$$

By above discussions and construction of the linear map $\phi$ is a Lorentzian reflection or composition of two Lorentzian reflections such that $\phi(u) = v$. ■

## 4. Lorentzian Rotations and Cartan - Dieudonné Theorem

**Theorem 7** *Product of two Lorentzian reflection matrices is a Lorentzian rotation matrix.*

**Proof:** Let $\varphi_{\mathbf{x}}$ and $\varphi_{\mathbf{y}}$ be two Lorentzian reflection matrices with the planes of reflection $x^{\perp}$ and $y^{\perp}$, respectively. Since $\det\varphi_{\mathbf{x}} = \det\varphi_{\mathbf{y}} - 1$, then $R = \varphi_{\mathbf{x}}\varphi_{\mathbf{y}}$ must have determinant 1. Now, we only need to show that $R^T I^* R = I^*$. By using the identities $\varphi_{\mathbf{x}}^T I^* = I^*\varphi_{\mathbf{x}}$, $\varphi_{\mathbf{y}}^T I^* = I^*\varphi_{\mathbf{y}}$, $\varphi_{\mathbf{x}}^{-1} = \varphi_{\mathbf{x}}$ and $\varphi_{\mathbf{y}}^{-1} = \varphi_{\mathbf{y}}$, we may write

$$R^T I^* R = (\varphi_{\mathbf{x}}\varphi_{\mathbf{y}})^T I^*(\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}) = {\varphi_{\mathbf{y}}}^T(\varphi_{\mathbf{x}}^T I^*)\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}$$
$$= ({\varphi_{\mathbf{y}}}^T I^*)(\varphi_{\mathbf{x}}\varphi_{\mathbf{x}})\varphi_{\mathbf{y}} = I^*\varphi_{\mathbf{y}}\varphi_{\mathbf{y}} = I^*.$$

■

**Theorem 8** *Any orthogonal transformation on $\mathbb{E}_1^3$ can be written as a decomposition of at most six Lorentzian reflections.*

**Proof:** We will apply Theorem 2 recurrently at each step. Let $R : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ be an orthogonal transformation and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthogonal basis for $\mathbb{E}_1^3$.
**Step 1:** First consider $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{u} = R(\mathbf{e}_1)$. Since $R$ is an orthogonal transformation, we have $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}}$. If $R(\mathbf{e}_1) \neq \mathbf{e}_1$, according to Theorem 2, there exists a linear map

$$\varphi_1(\mathbf{x}) = \begin{cases} \varphi_{\mathbf{e}_1}(\mathbf{x}) & \text{if } R(\mathbf{e}_1) = -\mathbf{e}_1 \\ \varphi_{R(\mathbf{e}_1)-\mathbf{e}_1}(\mathbf{x}) & \text{if } R(\mathbf{e}_1) \neq -\mathbf{e}_1 \text{ and } R(\mathbf{e}_1) - \mathbf{e}_1 \text{ is nonnull} \\ \varphi_{\mathbf{e}_1}\varphi_{R(\mathbf{e}_1)+\mathbf{e}_1}(\mathbf{x}) & \text{if } R(\mathbf{e}_1) \neq -\mathbf{e}_1 \text{ and } R(\mathbf{e}_1) - \mathbf{e}_1 \text{ is null} \end{cases}$$

where $\mathbf{x}_1 = R(\mathbf{e}_1) - \mathbf{e}_1$, that satisfies $\varphi_1(R(\mathbf{e}_1)) = \mathbf{e}_1$. Since $\varphi_1 \circ R$ is an orthogonal transformation, for $i = 2, 3$ we have that $\varphi_1(R(\mathbf{e}_i))$ is orthogonal to $\varphi_1(R(\mathbf{e}_1)) = \mathbf{e}_1$. Therefore

$$\varphi_1(R(\mathbf{e}_i)) \in Span\{\mathbf{e}_2, \mathbf{e}_3\}.$$

If it turns out that $R(\mathbf{e}_1) = \mathbf{e}_1$, then, according to Remark 3 we take $\varphi_1$ as the identity operator. We will do the same in the corresponding case of subsequent steps of the procedure.
**Step 2:** Now consider $\mathbf{v} = \mathbf{e}_2$ and $\mathbf{u} = \varphi_1(R(\mathbf{e}_2))$. If $\varphi_1(R(\mathbf{e}_2)) \neq \mathbf{e}_2$ then, application of Theorem 2 again yields the simple reflection

$$\varphi_2(\mathbf{x}) = \begin{cases} \varphi_{\mathbf{e}_2}(\mathbf{x}) & \text{if } \varphi_1(R(\mathbf{e}_2)) = -\mathbf{e}_2 \\ \varphi_{\varphi_1(R(\mathbf{e}_2))-\mathbf{e}_2}(\mathbf{x}) & \text{if } \varphi_1(R(\mathbf{e}_2)) \neq -\mathbf{e}_2 \text{ and } \varphi_1(R(\mathbf{e}_2)) - \mathbf{e}_2 \text{ is nonnull} \\ \varphi_{\mathbf{e}_2}\varphi_{\varphi_1(R(\mathbf{e}_2))+\mathbf{e}_2}(\mathbf{x}) & \text{if } \varphi_1(R(\mathbf{e}_2)) \neq -\mathbf{e}_2 \text{ and } \varphi_1(R(\mathbf{e}_2)) - \mathbf{e}_2 \text{ is non} \end{cases}$$

where $\mathbf{x}_2 = \varphi_1(R(\mathbf{e}_2)) - \mathbf{e}_2$, that satisfies $\varphi_2(\varphi_1(R(\mathbf{e}_2))) = \mathbf{e}_2$. Since $\varphi_1 \circ R$ is an orthogonal transformation, we have $\varphi_1(R(\mathbf{e}_1)) = \mathbf{e}_1$ is orthogonal to $\varphi_1(R(\mathbf{e}_2))$. Hence $\mathbf{e}_1 \in \mathbf{x}_2^{\perp}$ then $\varphi_2(\mathbf{e}_1) = \mathbf{e}_1$. Therefore, we have

$$\varphi_2(\varphi_1(R(\mathbf{e}_i))) = \mathbf{e}_i \text{ for } i = 1, 2$$
$$\varphi_2(\varphi_1(R(\mathbf{e}_3))) \in Span\{\mathbf{e}_3\}.$$

**Step 3:** Now consider $\mathbf{v} = \mathbf{e}_3$ and $\mathbf{u} = \varphi_2(\varphi_1(R(\mathbf{e}_3)))$. If $\varphi_2(\varphi_1(R(\mathbf{e}_3))) \neq \mathbf{e}_3$

then, application of Theorem 2 again yields the simple reflection

$$\varphi_3(\mathbf{x}) = \begin{cases} \varphi_{\mathbf{e}_3}(\mathbf{x}) & \text{if } \varphi_2(\varphi_1(R(\mathbf{e}_3))) = -\mathbf{e}_3 \\ \varphi_{\varphi_2(\varphi_1(R(\mathbf{e}_3)))-\mathbf{e}_3}(\mathbf{x}) & \text{if } \varphi_2(\varphi_1(R(\mathbf{e}_3))) \neq -\mathbf{e}_3 \text{ and } \varphi_2(\varphi_1(R(\mathbf{e}_3))) - \mathbf{e}_3 \text{ is nonnull} \\ \varphi_{\mathbf{e}_3}\varphi_{\varphi_2(\varphi_1(R(\mathbf{e}_3)))+\mathbf{e}_3}(\mathbf{x}) & \text{if } \varphi_2(\varphi_1(R(\mathbf{e}_3))) \neq -\mathbf{e}_3 \text{ and } \varphi_2(\varphi_1(R(\mathbf{e}_3))) - \mathbf{e}_3 \text{ is non} \end{cases}$$

where $\mathbf{x}_3 = \varphi_2(\varphi_1(R(\mathbf{e}_3))) - \mathbf{e}_3$. By construction of that satisfies $\varphi_3$, we have $\varphi_3(\varphi_2(\varphi_1(R(\mathbf{e}_3)))) = \mathbf{e}_3$. Since $\varphi_2 \circ \varphi_1 \circ R$ is an orthogonal transformation, $\varphi_2(\varphi_1(R(\mathbf{e}_1)))$ and is $\varphi_2(\varphi_1(R(\mathbf{e}_2)))$ are orthogonal to $\varphi_2(\varphi_1(R(\mathbf{e}_3)))$. Hence $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{x}_3^{\perp}$ then $\varphi_3(\mathbf{e}_1) = \mathbf{e}_1$ and $\varphi_3(\mathbf{e}_2) = \mathbf{e}_2$. Therefore, we have

$$\varphi_3(\varphi_2(\varphi_1(R(\mathbf{e}_i)))) = \mathbf{e}_i \text{ for } i = 1, 2, 3.$$

Thus we have

$$\varphi_3 \circ \varphi_2 \circ \varphi_1 \circ R = I_3.$$

Consequently, the orthogonal transformation $R$ can be expressed as the following product of at most six Lorentzian reflections:

$$R = \varphi_1^{-1} \circ \varphi_2^{-1} \circ \varphi_3^{-1} = \varphi_1 \circ \varphi_2 \circ \varphi_3.$$

■

**Remark 9** *If the number of reflections that decompose a given orthogonal transformation is even (odd) then the transformation is a Lorentzian rotation (reflection).*

## Applications

**Example 10** *Let us find the Lorentzian reflection matrix with the plane of reflection*

$$\mathbf{x}^{\perp} = \{(y_1, y_2, y_3) \in \mathbb{E}_1^3 : y_1 - y_2 + 2y_3 = 0\}.$$

*Here* $\mathbf{x} = (-1, -1, 2)$. *By using Theorem 2, we find*

$$\varphi_{\mathbf{x}} = \begin{bmatrix} 3/2 & -1/2 & 1 \\ 1/2 & 1/2 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

**Example 11** *Let us find the plane of reflection for the Lorentzian reflection matrix*

$$\varphi = \begin{bmatrix} 3/2 & -1 & -1/2 \\ 1 & -1 & -1 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

*By long and tedious calculations, we find the eigenvector of $\varphi$ corresponding to the eigenvalue $-1$ as $\mathbf{x} = (1, 2, 1)$. So the plane of reflection is found as*

$$\mathbf{x}^{\perp} = \{(y_1, y_2, y_3) \in \mathbb{E}_1^3 : -y_1 + 2y_2 + y_3 = 0\}.$$

**Example 12** *Consider the spacelike vectors $\mathbf{u} = (-1, 1, -1)$ and $\mathbf{v} = (7, 5, -5)$ such that $\mathbf{u} \neq \mathbf{v}$ and $\varepsilon_{\mathbf{u}} = \varepsilon_{\mathbf{v}} = 1$. By Theorem 2 there is a linear map $\varphi : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ such that $\varphi(\mathbf{u}) = \mathbf{v}$. Since $\mathbf{v} \neq -\mathbf{u}$ and $\mathbf{u} - \mathbf{v} = (-8, -4, 4)$ is nonnull, then the linear map $\varphi$ is defined as*

$$\varphi(\mathbf{x}) = \varphi_{\mathbf{u}-\mathbf{v}}(\mathbf{x}).$$

*By long and tedious calculations, we find linear map $\varphi$ as follows:*

$$\varphi(\mathbf{x}) = (-x_1 + 2x_2 - 2x_3, x_1 + 2x_2 - x_3, 2x_1 - x_2 + 2x_3).$$

*The matrix representation of linear map $\varphi$ is*

$$\varphi = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}.$$

*It is easily seen that $\varphi(\mathbf{u}) = \mathbf{v}$ and $\varphi$ is a Lorentzian reflection.*

**Example 13** *Consider the following orthogonal transformation*

$$\begin{aligned} &R : \mathbb{E}_1^3 \to \mathbb{E}_1^3 \\ &\quad : R(x_1, x_2, x_3) = (\frac{9}{4}x_1 - 2x_2 + \frac{1}{4}x_3, x_2 - x_1 - x_3, 2x_2 - \frac{7}{4}x_1 + \frac{1}{4}x_3) \end{aligned}$$

*which has matrix representation as*

$$R_{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}} = \begin{bmatrix} 9/4 & -2 & 1/4 \\ -1 & 1 & -1 \\ -7/4 & 2 & 1/4 \end{bmatrix}.$$

*Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{E}_1^3$.*
***Step 1:*** *First consider $\mathbf{v} = \mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{u} = R(\mathbf{e}_1) = (9/4, -1, -7/4)$. Since $R(\mathbf{e}_1) \neq \mathbf{e}_1$, then there is a linear map $\varphi_1$ such that $\varphi_1(R(\mathbf{e}_1)) = \mathbf{e}_1$*

*according to Theorem 2. And* $R(\mathbf{e}_1) \neq -\mathbf{e}_1$ *and* $R(\mathbf{e}_1) - \mathbf{e}_1 = (5/4, -1, -7/4)$ *is nonnull. By Theorem 8, the linear map* $\varphi_1$ *is defined as*

$$\begin{aligned}\varphi_1(\mathbf{x}) &= \varphi_{R(\mathbf{e}_1)-\mathbf{e}_1}(\mathbf{x}) \\ &= \varphi_{(5/4,-1,-7/4)}(\mathbf{x}) \\ &= (\frac{9}{4}x_1 + x_2 + \frac{7}{4}x_3, -x_1 + \frac{1}{5}x_2 - \frac{7}{5}x_3, -\frac{7}{4}x_1 - \frac{7}{5}x_2 - \frac{29}{20}x_3)\end{aligned}$$

*with the matrix representation*

$$\varphi_{1\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}} = \begin{bmatrix} 9/4 & 1 & 7/4 \\ -1 & 1/5 & -7/5 \\ -7/4 & -7/5 & -29/20 \end{bmatrix}.$$

*Here* $\varphi_1(R(\mathbf{e}_2)) = (0, -3/5, -4/5)$ *and* $\varphi_1(R(\mathbf{e}_3)) = (0, -4/5, 3/5)$. *That is*

$$\varphi_1(R(\mathbf{e}_i)) \in Span\{\mathbf{e}_2, \mathbf{e}_3\}$$

*for* $i = 2, 3$.

***Step 2:*** *Now consider* $\mathbf{v} = \mathbf{e}_2 = (0, 1, 0)$ *and* $\mathbf{u} = \varphi_1(R(\mathbf{e}_2)) = (0, -3/5, -4/5)$. *Since* $\varphi_1(R(\mathbf{e}_2)) \neq \mathbf{e}_2$ *then there is a linear map* $\varphi_2$ *such that* $\varphi_2(\varphi_1(R(\mathbf{e}_2))) = \mathbf{e}_2$ *according to Theorem 2. We know that* $\varphi_1(R(\mathbf{e}_2)) \neq -\mathbf{e}_2$ *and* $\varphi_1(R(\mathbf{e}_2)) - \mathbf{e}_2 = (0, -8/5, -4/5)$ *is nonnull. Application of Theorem 8 yields the linear map* $\varphi_2$ *is defined as follows:*

$$\begin{aligned}\varphi_2(\mathbf{x}) &= \varphi_{\varphi_1(R(\mathbf{e}_2))-\mathbf{e}_2}(\mathbf{x}) \\ &= \varphi_{(0,-8/5,-4/5)}(\mathbf{x}) \\ &= (x_1, -\frac{3}{5}x_2 - \frac{4}{5}x_3, -\frac{4}{5}x_2 + \frac{3}{5}x_3).\end{aligned}$$

*Here the matrix representation of the Lorentzian reflection is obtained as*

$$\varphi_{2\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix}.$$

*Moreover, we get*

$$\varphi_2(\varphi_1(R(\mathbf{e}_i))) = \mathbf{e}_i$$

*for $i = 1, 2, 3$. Thus, the orthogonal transformation $R$ can be expressed as decomposition of two simple Lorentzian reflections:*

$$R = \varphi_1 \circ \varphi_2$$

*that is*

$$R = \begin{bmatrix} 9/4 & -2 & 1/4 \\ -1 & 1 & -1 \\ -7/4 & 2 & 1/4 \end{bmatrix} = \begin{bmatrix} 9/4 & 1 & 7/4 \\ -1 & 1/5 & -7/5 \\ -7/4 & -7/5 & -29/20 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix}.$$

*Consequently, $R$ is a Lorentzian rotation.*

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