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RESEARCH

Geometric Figures in the Facades of the Khoja Akhmet Yasawi Mausoleum

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Abstract

The purpose of the current study was to comprehensively investigate the facades of the Mausoleum of Khoja Akhmet Yasawi, its geometric patterns and the mathematical laws used for their design. To do this, the writings of medieval mathematicians were analysed, and their principles considered to model the geometric patterns of the facades. The paper describes the evolution of geometric laws in the Islamic world from the 7th to the 9th century. The Mausoleum of Khoja Akhmet Yasawi is one of the best examples of geometric architectural applications from the 14th-15th century. The facades of Khoja Akhmet Yasawi seemingly rely on the methods of two mathematicians, Al-Farabi and Al-Khwarizmi. To a greater extent, the ornaments derive from dividing equilateral triangles, parallelograms and polygons to create geometric figures. The façade patterns of the Khoja Akhmet Yasawi complex require careful calculation and preparation due to their conciseness and harmony, giving insight into the fields of arithmetic, geometry and architecture and architects' education during the 14th-15th century.

Keywords Symbolism · Middle Ages · Islamic World · Grandiose Building · Doctrine of Harmony · Architectural Methods

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Introduction

The Mausoleum of Khoja Akhmet Yasawi is an important cultural heritage site of Central Asia, attracting attention with its exquisite architecture and rich ornamentation. This mausoleum, dedicated to the Sufi mystic Khoja Akhmet Yasawi, not only symbolises religious and cultural values but is also an object of study for researchers interested in geometry and art. The research provides an opportunity to study the writings of Amir Temir and what influence the activities of Al-Farabi and Al-Khwarizmi, as prominent scholars and philosophers of the classical era (650–1250). The geometric laws used in the patterns of the complex facades can encourage mathematicians, architects, artists, and other related figures to make discoveries. In recent years, Central Asians have become increasingly attracted to the history and culture of the region (Lubin 1995; Berggren 2014), as seen among residents and travellers who want to learn more about Kazakh culture, creativity and architecture.

Bonner (2017) has comprehensively examined the polygonal technique, the cornerstone method used by master artists to create complex Islamic geometric patterns. Bonner described the design methodologies that have preserved this nearly lost art form, supported by historical evidence that emphasizes its evolution across cultures from the Taj Mahal to the Mediterranean. By examining the key role of geometry in Islamic ornamentation, including its integration with calligraphy, floral motifs, and architectural elements, the study emphasizes the enduring significance of this technique.

The Khoja Akhmet Yasawi complex has colossal dimensions and exceptional architecture, compared with similar buildings of the Timurid period (14th-15th centuries) and, according to researchers, embodied all the achievements of the construction art of that time (Mustapaeva 2022a, b). The complex has a pointed dome, the largest one in Kazakhstan and Central Asia. These facts highlight the great cultural and historical significance of the mausoleum, which further reinforces the relevance of this study. As described by Samoilov et al. (2023) Khoja Akhmet Yasawi, as a historical monument, requires some kind of restoration intervention, with a choice between a complete restoration of the complex, or, the conservation of the mausoleum in its current form. The first will restore its former beauty and access for tourists whereas the second will preserve its historicity.

During the 7th-14th centuries, the use of geometric patterns in significant buildings, such as mosques and mausoleums, was a feature of the Arab Caliphate. The ornaments could be of different shapes, using different forms and construction methods, but could also give a sense of movement and dynamism of the architecture due to the play of natural light and shadows. Architects and mathematicians who lived in Islamic countries made a decisive contribution to the creation of a new, unique style (King 1983). Kemelbekova et al. (2023) described the grandiose dimensions of Khoja Akhmet Yasawi and mentioned that according to the ruler of the Golden Horde Temerlan's plan, the mausoleum was built on the basis of surahs of the Holy Quran, to spread Islam in Central Asia. Researchers also described the interior, however, they did not pay due attention to the geometry and symbolism of the facade patterns (The contribution of… 2016).

Cromwell (2010) conducted a mathematical analysis of the Topkapi Scroll, a key document for the study of Islamic geometric ornamentation. It examines exemplary ornaments based on a modular construction that utilizes six fundamental tiles to create complex patterns. Cromwell explored various applications of this system, including the integration of non-standard elements, the use of Penrose rhombuses for non-periodic patterns, and the development of multiscale bi-level designs that simultaneously display complementary patterns at different scales. The study compares different methods for creating large-scale and small-scale patterns, emphasizing the versatility and complexity achievable through tile breakdown and modular techniques. By analyzing specific panels from the Topkapi Scroll, the study demonstrates how these geometric principles are used in architectural contexts, especially in Iranian structures, showcasing the enduring sophistication of Islamic geometric art.

While previous studies have described the mausoleum's architectural features, there has been limited investigation into the specific geometric principles underlying its decorative elements and their connection to medieval Islamic mathematical thought. The Mausoleum of Khoja Akhmet Yasawi serves as an important cultural symbol, reflecting the spiritual and artistic aspirations of the era. Its preservation is crucial not only for maintaining the integrity of Kazakhstan's architectural landscape but also for understanding the historical significance of Islamic mathematics in shaping architectural practices (Bidaibekov et al. 2016). The ongoing debates regarding its conservation whether to restore it to its original grandeur or to preserve its current state highlight the complex relationship between heritage preservation and historical authenticity. Thus, the study of the Khoja Akhmet Yasawi Mausoleum presents an opportunity to broaden implications for heritage conservation in Central Asia.

The purpose of the study was to analyse the geometric patterns in the Mausoleum of Khoja Akhmet Yasawi through an investigation of the mathematical methods and principles developed by Al-Farabi and Al-Khwarizmi that were applied in its construction. Specifically, this study seeks to:

- document and analyse the geometric laws and construction methods used in the facade patterns of the mausoleum;
- examine how the mathematical theories of Al-Farabi and Al-Khwarizmi were practically implemented in the architectural design;
- investigate the relationship between geometric patterns and their symbolic meanings in Islamic architectural traditions of the 14th-15th centuries.

Several methodological approaches were considered to address these research questions. Computer-aided design (CAD) analysis and 3D modeling could potentially provide precise geometric measurements and pattern generation algorithms. Archaeological investigation of construction techniques might reveal physical evidence of building methods. However, these approaches were deemed less suitable for our primary research aims. CAD analysis, while precise, risks imposing modern mathematical frameworks on historical designs. Archaeological investigation, though valuable, is limited by preservation requirements and access restrictions to the historical structure. Instead, we adopted a combined approach of historical document analysis, architectural examination, and geometric pattern analysis. This methodology was chosen

for its ability to bridge theoretical mathematical principles with practical architectural implementation, though it presents certain limitations. The significant temporal gap between the mathematical texts and the mausoleum's construction makes it challenging to establish direct links between theory and practice. Manual modeling of patterns, while maintaining historical authenticity, may not capture the full precision of original construction methods.

The significance of this study lies in its attempt to understand how medieval Islamic mathematical knowledge was transformed into architectural practice, a process that remains poorly documented in current scholarship. While previous studies have described the mausoleum's architectural features (Mustapaeva 2022a, b) or analyzed Islamic geometric patterns generally (King 1983), none have systematically investigated the relationship between theoretical mathematical principles and their architectural implementation in this specific context. The study focuses solely on external facade patterns, excluding interior designs and structural engineering aspects. This limitation was necessary to maintain analytical depth within the scope of our research aims, though it leaves room for future studies exploring other aspects of the mausoleum's mathematical and architectural features.

Materials and methods

While working on the current study, a detailed historical analysis of documents, archival data, literature on architecture, and the arts of the 14th-16th centuries was carried out to discern the context, the use, and symbolism of the Khoja Akhmet Yasawi patterns (Mustapaeva 2022a, b). In particular, the book by Fergusson (2022) *The History of Modern Architectural styles* and a paper by Zeina (2022) on the Golden Ratio were investigated. The biography of the ruler Amir Temir, the mathematician-philosopher Al-Farabi, and the astronomer-mathematician Al-Khwarizmi were studied to understand the geometric laws and motifs, which they wanted to capture in patterns during the construction of the mausoleum (King 1983; Issin and Romazanova 2024).

The architectural analysis included the investigation of the architectural elements of the complex, geometric proportions, the location of geometric patterns and other details on the facades of the building to identify the features and patterns in the design of the mausoleum. The geometric analysis was carried out to study the geometric methods, laws and structure of the facade patterns, which guided the architects and builders in the construction of the Khoja Akhmet Yasawi Mausoleum. It included an analysis of symmetry, repeating elements, proportions of patterns, and their angles and aspect ratio. The specific methods of dividing squares, triangles, and equilateral polygons proposed by mathematicians Al-Farabi and Al-Khwarizmi were analysed and described. The patterns of the main facade and the facade of the walls were manually modelled. Visualization of the simulation and its description are also proposed in the paper. This helped to identify specific mathematical laws used in the design of geometric patterns in the Mausoleum of Khoja Akhmet Yasawi.

The analysis of the geometric patterns of the facades and architecture of Khoja Akhmet Yasawi was carried out comparing them with other historical and architectural objects erected both in Central Asia during the Middle Ages and in other regions

and historical periods. The analytical comparison considered monuments and structures from Europe (e.g., the Netherlands, Turkey) and Asia (e.g., Kazakhstan, Iran). This helped to identify common features of the geometric patterns use in different temporal and cultural contexts and determine the architectural style specific to the mausoleum.

An analytical study of the pattern's symbolic meanings, inscriptions located between them, and their connection with the religious and cultural traditions of the region was conducted to understand the value of the Mausoleum from a cultural and religious perspective. As a result, the conclusions were analysed and systematised to better understand the context, aiding further investigations of the geometric laws and methods used in the patterns of other architectural monuments.

Results

The mosque-mausoleum of Khoja Akhmet Yasawi, located in the south of Kazakhstan in the city of Turkestan, was built on the site of a 12th-century mazar, where the ashes of the Sufi preacher and poet Khoja Akhmet Yasawi (1105–1166) were kept by decree of Emir Timur. Originally conceived as a dervish abode or khanaka, the mausoleum became an architectural treasure and an indicator of the architects and craftsmen's abilities (Mustapaeva 2022b; Scott 1879; Pasha et al. 2020). The development of art and architecture in medieval Central Asia was closely linked to the concept of equilibrium. The doctrine of balance was part of the geometric harmony doctrine used in various architectural structures and methods of their organisation. Scholars and encyclopaedists of the Islamic Middle Ages noted the aesthetic meaning in the teachings of harmony and balance. They also saw them as the basis of practical art. Geometric methods became an important ideological basis for architecture, and the development of architecture in medieval Central Asia (Stanford Encyclopedia 2020). The architectural and artistic creativity of medieval artisans specializing in geometric patterns became the basis for the establishment of a practice associated with a new interpretation of mathematical sciences. New theoretical principles emerged based on practical experience, including construction and art.

It was important to consider the entire architectural structure of buildings, balance, stability, and strength. The research of medieval mathematicians studying geometric methods on which the architecture was based had applications in mathematics. The main and most significant part of these studies was algebra, which defines unknown geometric and numerical questions. Geometric patterns and ornaments became part of the architectural decor and spiritual symbols, arousing interest and respect for the teachings of balance and harmony (Kerimkhulle et al. 2023). The research of mathematicians of that time found a response in the minds of ordinary people. Al-Farabi, whose works influenced the view of Khoja Akhmet Yasawi to a greater extent, appreciated architecture and many other practical arts. He mentioned that many masters of his time used geometric methods, including the art of construction management, project implementation, and architectural construction. Al-Farabi expressed the opinion that geometric foundations are the basic knowledge of the master and their study should begin with geometric methods. He highlighted that the experimental art is the

basis of geometric methods and their application in relation to figures, bodies, intentions, positions, and order (Jantayev 2024).

Among the works of the astronomer Al-Khwarizmi is the *Small Book on Algebra and Muqabala Problems* (van Dalen 2014). The demand in the spheres of the caliphate's economy, domestic and foreign trade, agriculture, irrigation construction (urban construction, architecture, irrigation) influenced the emergence of algebraic calculations since geometric methods were almost vital for society. The preface of Al-Khwarizmi's treatise mentions that he compiled a small book on algebra and calculations, including simple and complex arithmetic questions, to aid people's inheritances, wills, distributions, lawsuits, trade, land management, canal construction, geometry (engineering, construction of architectural objects) (Bidaibekov et al. 2023). Al-Khwarizmi described three types of numbers in algebra consisting of squares of numbers, roots, and primes. In his treatises, the scholar considered a square as a unit of surface measurement, provided rules for calculating the areas of triangles and equilateral parallelograms (square, rhombus), and gave information about regular triangles and their sides (Fernández-González 2019). He also presented rules for calculating the areas of squares, triangles, and rhombuses (Issin and Romazanova 2024).

Algebra, particularly the work of scholars like Al-Farabi and Al-Khwarizmi, laid the groundwork for geometric principles that were applied in architectural practices during the medieval period. Al-Khwarizmi's contributions provided measurements essential for construction. His methods involved algebraic calculations vital for urban planning, irrigation, and architectural design, which directly impacted the construction techniques employed in monumental structures such as the Khoja Akhmet Yasawi Mausoleum. The mausoleum's elaborate geometric patterns reflect the theoretical frameworks established by these mathematicians. The use of proportions, symmetry, and geometric harmony, as espoused by Al-Farabi, can be observed in the intricate designs of the mausoleum's facades. Furthermore, the symbolic meanings embedded in the patterns of the mausoleum are indicative of the cultural significance of algebra and geometry in Islamic architecture. The blending of spiritual and mathematical elements in the mausoleum exemplifies how algebraic principles were not merely theoretical but played a crucial role in shaping the architectural identity of Central Asia.

During the 14th-15th centuries in Central Asia, the leading architects were perceived as educated, intelligent workers who had mastered mathematical sciences. Therefore, during the design and construction of important structures as mausoleums and mosques, architects wanted to attract the best specialists of geometric techniques (Mehdi et al. 2021). In the Middle Ages, a highly qualified architect was educated through practical experience, namely by participating in construction and getting knowledge from colleagues on site. In some cases, it was possible to get specialised books related to the construction profession.

Khoja Akhmet Yasawi was originally a propagator of Sufi teachings in Central Asia. However, ruler Amir Temir, wanting to show the wealth and glory of his country and to promote the Muslim religion, based the building on the Holy Quran and made it the largest brick complex in the region. In addition to the construction methods adopted at that time, Amir Temir paid great attention to the use of more modern

technologies, both in construction and architecture. He attached great importance to the decor of the building and its basic dimensions. Some examples include cauldrons, different shapes of cast candles, different-sized doors with wooden carvings decorated in gold and silver, ceramic stickers in different colours and inscriptions on facades. Some architectural features of the Karakhan period and elements of Kazakh ornament can be found in the complex (The contribution of… 2016). The two basic rules used in geometric patterns on the facades of the Khoja Akhmet Yasawi Mausoleum are as follows:

1. The theoretical consideration of irrational relations $1:\sqrt{2},\ 1:\sqrt{3},\ 1:\sqrt{5}$, and correspondences and accurate records between rectangles depicted on a radial coordinate grid;
2. The method of transferring a drawing to nature, based on simultaneously acting proportions (Lubin 1995).

The use of mathematical methods by masters of ornament was often underestimated by researchers. However, based on the study of mathematical treatises, the understanding of the architectural aesthetics of Medieval Islam has significantly expanded. In his treatise, Abu Nasr Muhammad bin Tarkhan bin Ausalag Al-Farabi (870–950) presented two ways of constructing simple geometric shapes using a parabola, circle, straight line, trisection, and angle multiplication. One of the books of the figure is entirely devoted to squares and rhombuses. It shows the methods which allow fitting an octagon into a square (Fig. 1).

The diagonal lines of the square are drawn in the square ABCD and the point of their intersection is marked as E. To determine point G, from point E, along the diagonal AD, half the length of the edge of the square is measured. A straight line

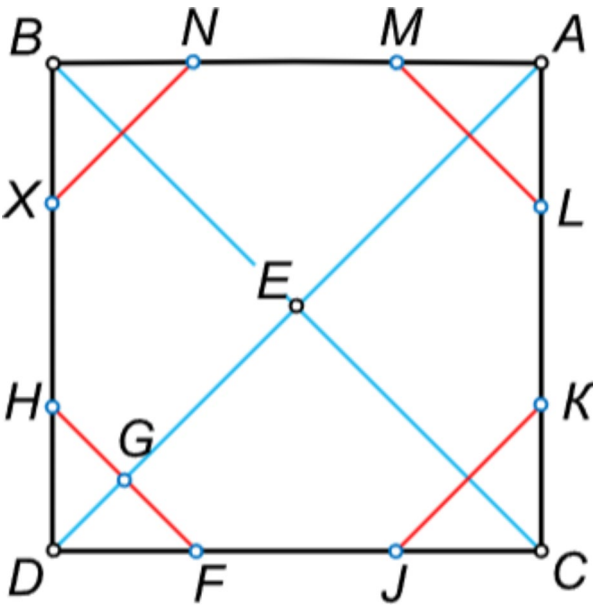


Fig. 1 Division of a square into an equilateral octagon by Al-Farabi. Image: Authors

is drawn perpendicular to the diagonal of the square through point G. Intersecting with the sides of the rectangle, it forms the points H, F. When such manipulations are repeated, points X, N, M, L, K, J are also formed on the sides of the square. These points form an equilateral octagon HFXNMLKJ inscribed in the square ABCD. In addition, this method divides the shape into four triangles with equal sides.

The astronomer-mathematician Al-Khwarizmi lived in the 9th century in the north of the Arab Caliphate and was well-trained in ancient Indian mathematics. His creative works go back to the traditions of ancient Babylon. In his book *An arithmetic treatise. The book on addition and subtraction*, the mathematician distinguished three types of existing numbers used for calculations in algebra. He divided them into roots, squares, and prime numbers. A root is any number, be it a small fraction, a unit, or a large number that can be multiplied by itself. It is created by multiplying the square, and prime numbers are numbers that are not related to the root or square. Translating this into geometry, the square is the unit of measurement of the surface, and the root is the side of the square. The information given by Al-Khwarizmi on the division of whole regular triangles with sides 3:4:5, 5:12:13, 9:12:15. The shown methods for calculating the areas of rhombuses, triangles and squares inspired Amir Temir and his architects to create similar patterns. Figure 2 shows a special case of proving the theorem about the sum of squares of edges equal to the square of the hypotenuse.

Perhaps the scholar wrote his books specifically for architects, since expressions 'such a column has a round base,' 'pyramidal column' related to the art of architecture are found in them, and architectural objects are most often cited as examples of problem-solving. Interestingly, Al-Khwarizmi proposed to define irrational roots such as $\sqrt{50}$ and $\sqrt{75}$ as prime numbers and determine their values only approximately. Perhaps this is conditioned by the need for a module to help builders and

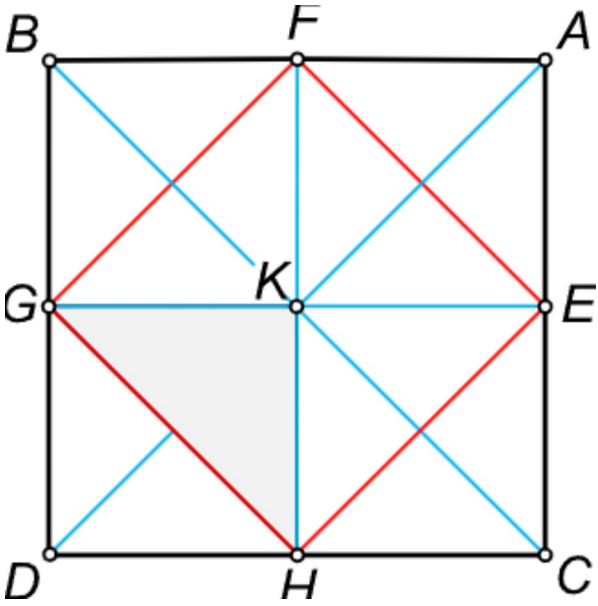


Fig. 2 Al-Khwarizmi's proof about the sum of squares of edges equal to the square of the hypotenuse theorem. Image: authors

architects since the size of bricks was not fixed and varied from region to region at that time. Although the modular grid took into account the thickness of the seam, due to manual work, its arithmetic calculation was disrupted by the instability of the work. These inconsistencies and errors were corrected in various geometric ways with ropes, rulers, and poles. Hence, the idea of inconsistencies and errors in the application of geometric, arithmetic, and modular relations appeared.

The ruler Amir Temir, wanted to strengthen the ideological policy and spread Islam. To do this, he built a new Khoja Akhmet Yasawi complex to modernise it in accordance with modern culture. The mausoleum, like other buildings of the ruler, differed in its size and magnitude of the design, although the implementation of the three-dimensional idea was simple. It is presented in the form of two domes and a powerful portal in an elongated parallelepiped (The contribution of… 2016). The facades of the mausoleum are rich in geometric ornaments that have important spiritual and symbolic meanings in the architecture and art of Islam. The geometric patterns of Islamic culture are based on certain mathematical principles reflecting the original way of the universal order (Bugaevsky et al. 2020). In the Khoja Akhmet Yasawi Mausoleum, the facades were decorated with geometrically complex patterns based on sacred geometry, which use both polygons and circles. These patterns were based on basic motifs, which gradually expanded due to the repetitions of its parts. The main facade of the building faces south-east, just as the Kazakhs built their yurts in ancient times. It consists of a main large portal and two towers, on which there are three entrance doors and the main entrance arch. The side facade depicts a series of polygons and stars used to create a holistic complex pattern. They are decorated with burnt bricks of different colours (Berggren 2014).

The right side of the mausoleum is turned to the northeast. It consists of a mosaic laid out of dark blue brick and two diamond-shaped octagonal elements laid out in two horizontal and two vertical rows. Such a brick consists of smaller square bricks laid at an angle of 45°. On this wall, there are the words ‘Allah,’ ‘Muhammad,’ ‘O Lord,’ ‘Power belongs to God,’ ‘Allah is my ruler’ laid out of light blue bricks. The method of building such a dark blue brick is quite simple (Fig. 3).

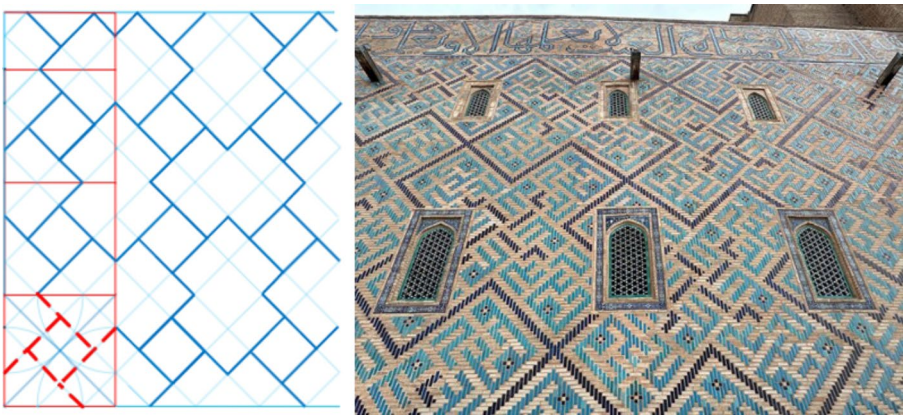


Fig. 3 Diagram of the facade pattern of the right wall of the building. Image: Authors

The diagonals are drawn in the ABCD square. Their intersection defines the point O. Arcs of radius R (R=AO) are drawn from this point, with the help of which points E and G are determined. These points divide the side of the square into three parts. Arithmetically, they can be written as:

$$\mathrm{DG} = \frac{2-\sqrt{2}}{2};\ \mathrm{GE} = \sqrt{2}-1;\ \mathrm{EC} = \frac{2-\sqrt{2}}{2}$$

.

If the points E and G are connected, four lines parallel to the diagonals are formed. These lines are adjacent to the sides of the square at an angle of 45° and form four equilateral triangles with them, which, in turn, intersect and form a smaller equilateral square, the sides of which are located at an angle of 45° to the sides ABCD. From this geometric pattern, a mosaic is created when the sides of a small square are extended to the E points. In this geometric pattern, calligraphy is used to add text to the overall design, which gives a sense of these two types of important art for Islam. The left side of the building faces southwest. There are two rows of twelve terminal cross-shaped elements where a swastika can be visible (Fig. 4). And inside these geometric elements are inscribed the words 'Allah,' 'Muhammad,' 'O Lord,' 'Power belongs to God,' 'Allah is my ruler' as well as on the right side, which reflects the importance of symmetry.

The mosaic of the facade of the left wall is created from the same geometric pattern that is used for the right one, however, here, a shape G (sides AB) HE (sides DC) is used, extending outside the square area from point E (sides DC) through point G (sides AC) and symmetrically mapped through sides AB and AC to create the pattern. Based on the correct calculations and the location of the mosaic of the complex foundation, due to the light and shadows, a sense of the dynamics of the pattern is created throughout the day. This gives the design an overall complexity and beauty. The mosaic consists of a square twelve-point geometric figure, the feature of which is the same length of eight sides, and the remaining four are a quarter of the length. The octagon in the form of a star is located in the centre of the pattern, and the angles



Fig. 4 Geometric structure of the mosaic on the left façade. Image: Authors

of its rays are 72°. To construct it, one of the four ways of dividing a square described by Al-Farabi (Fig. 5) was used.

Due to the described method of constructing the facade pattern on the right wall of the mausoleum, points E and G divide the sides of the square into 3 parts. When points E and G are connected by an arc from different sides, points F are formed at the intersection with the diagonal. Connecting these points to the vertices of ABCD, the lines at the intersection form the points H, and connecting them to F, an equilateral octagon appears inside the square. At the intersection of the lines FE and FG, a point M appears, which results in 4 equilateral triangles MGE. At the intersection of diagonals and lines FG, FE, points K are formed, connecting them with M and O. A quadrangular star is obtained (the central part of the pattern) dividing the octagon into four quadrangles (the inner corners of the pattern). Straight lines drawn from the vertices of the square to the vertex of these quadrangles, at the intersection with the continuation of the lines AH, BH, CH, DH, form the last necessary point for eight quadrangles (the outer corners of the pattern). And the equilateral hexagons that have appeared are the inner, empty part of the ornament.

Discussion

The Mausoleum of Khoja Akhmet Yasawi is a manifestation of the architectural art in Medieval Central Asia which used various mathematical laws to create a specific appearance. The conducted research confirms that the patterns and ornaments on the facades of the mausoleum both have an aesthetic function and reflect the deep mathematical concepts described by the mathematical philosophers Al-Farabi and Al-Khwarizmi. According to Al-Farabi, geometry and arithmetic are included in all sciences of art. When using mathematical sciences to construct architectural forms, the technique of artisan mathematicians in architecture generates the doctrine of geo-

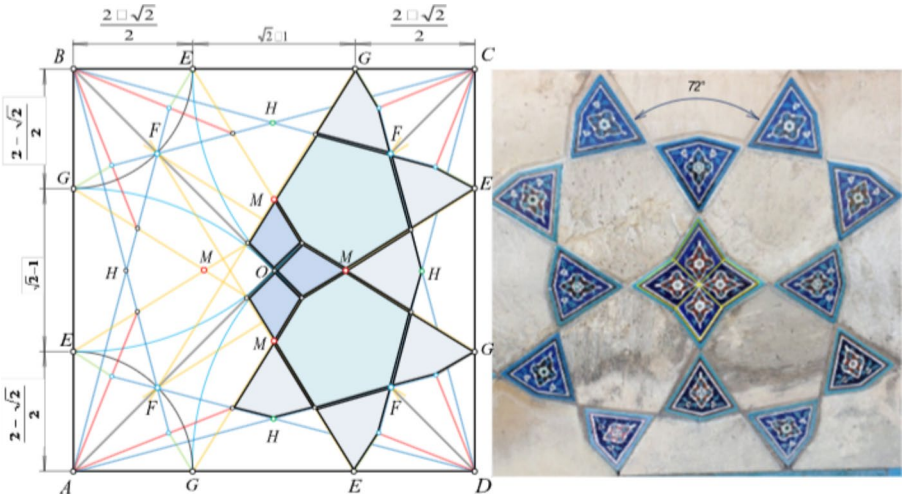


Fig. 5 Equilateral octagon on the foundation. Image: Authors

metric harmony (Brovko et al. 2024). He believed that concepts created as a generalisation of experience influenced the worldview of Medieval Asian decorative artists and architects. The scope of applied mathematical methods was much larger and included problems of architectural practice and theory, such as the balance of architectural masses, the structure of the building space, and fragments of architectural ornaments.

The mausoleum is often described as a style of geometric and floral compositions. Following its example, various monuments of the Karakhanid era were erected, such as the mausoleum of Ibrahim ibn Hussein, the public baths of Shahjuvara, and numerous Uzbek mausoleums. The main facade of Khoja Akhmet Yasawi is decorated with carved terracotta panels lined with paired bricks, and the patterns consist of round rosettes with cross-shaped spaces between them which are filled with ornaments that resemble vegetable ligature. The layout of the complex belongs to the facade single-chamber mausoleums, common for Southern and Northern Uzgen mausoleums, with doorways in all four directions.

Visual harmony and symmetry in the geometric patterns of Khoja Akhmet Yasawi are revealed, which indicates the explicit use of the doctrine of geometric harmony and balance. It confirms the hypothesis that medieval architects, builders, artists who worked on the complex were familiar with the principles of geometric methods and used them to achieve the functional and aesthetic integrity of the mausoleum (Bidaybekov et al. 2017). There are not many detailed studies on the geometric methods used in the construction of the Khoja Akhmet Yasawi Mausoleum, however, it can be assumed that the artists, builders, and architects of medieval Central Asia had extensive knowledge of the field of geometry and applied them to the design of buildings. Likely, the architects used the geometric laws described by Al-Farabi and Al-Khwarizmi, but also special geometric proportions based on Fibonacci numbers or the golden ratio to create a harmonious and aesthetic design, and calculate structural elements to ensure the strength and durability of the complex (Mao et al. 2024).

However, the Khoja Akhmet Yasawi Mausoleum is not the only indicator of medieval builders and architects' skill. As described in the journal Architectural Histories, during the 14th-15th centuries, a class of highly qualified artists appeared in the Netherlands and Europe, who also often engaged in architectural projects using geometric methods (Kik 2024). As the researchers explain, the emergence of such niche geometric practices is caused by the influx of humanistic and Italian views, for example through the works of Vitruvius or Alberti. Their treatises were among the first in the field of architecture when treatises appeared during the Italian Renaissance (Harris-McCoy, 2024). These facts indicate that the art of using geometric laws among the architects of the Middle Ages developed simultaneously all over the world, however, differences in culture and religion influenced the final works in different ways.

A good example of historical and architectural memory is the Hagia Sophia in Constantinople (Istanbul). In 1453, the Turkish Sultan Mehmed II conquered the cathedral and, amazed by its beauty, decided to turn the temple into a mosque. The cathedral remained mostly undamaged, but changes were required such as the addition of four minarets (Oikonomou 2022; Ar 2020). These historical events influenced studies on the cathedral. As noted by Savvides (2024), the architecture of the main building required careful research in various fields to clarify the geometric relation-

ships of the building. A full understanding of the geometrically complex design of St. Sophia Cathedral remains unattainable due to its inherent problems of creating a single model by integrating various geometric patterns. In turn, Agirbas (2020) investigated the construction of star-shaped polygons depicted on the slabs at the Akhlat cemetery. Its polygons differ significantly from those found at Khoja Akhmet Yasawi since they were built according to different geometric methods.

An example showing the difference between cultures and methods of constructing architectural patterns is the monument on the Costa Barrow, in Amphipolis. Due to the good condition of the monument, it is possible to qualitatively discern the geometric proportions and architectural values of the Macedonian people who built these tombs. According to Savvides (2023), the architects presumably relied on the principles of ad-quadratum (based on a square and defines proportions) and octature (based on an octagon and adheres to the ratios 1:2), and the lion most likely symbolizes those buried (Chugg 2021). Duvernoy (2021) and Rüstem (2019) described the evolution that Baroque architecture underwent. At this time, architects began to depart from the accepted rules of building construction as the application of curved spaces and wavy walls shows. The Baroque embodied architectural change, and in particular, resulted from increased education and technological developments that allowed erecting such buildings (Bieliatynskyi et al. 2022). These prove that for many centuries, before and after the construction of the Khoja Akhmet Yasawi Mausoleum, architects around the world used geometric methods and ratios for their architectural projects and symbolism.

Important for the study of architectural and historical heritage is the problem of its preservation. In the case of Khoja Akhmet Yasawi, it is necessary to select and summarise suitable materials. The study by Parlewar (2022), using the example of an old dining room, shows methods to reconstruct the interior of a building according to hexagonal geometric patterns. Milosz et al. (2020), use the example of the Sher-Dor Madrasa located on Registon Square in Samarkand, Uzbekistan. They described the method for restoring Timurid mosaics while preserving its historical and aesthetic value.

Fathi and Asgari (2023) analysed the change in geometric proportions and aspect ratios in the architecture of Imam Khomeini City Square in Tehran during three key historical periods, namely, the period of Nasseri, Pahlavi I, and Pahlavi II. The first period was dominated by proportions $\sqrt{1}$, $\sqrt{5}$, $\sqrt{7}$ and aspect ratios $\sqrt{2}$ for government buildings, theatres, museums, and street entrances such as Nassery and Bob Homayon. During the Pahlavi I period, the influence of the golden section in Iranian architecture began, which led to an increasing use of Iranian-Islamic proportions, such as $\sqrt{2}$ and $\sqrt{3}$. In the period of Pahlavi II, with the advent of modernism, specific proportional applications mostly disappeared. They were becoming more random and less systematic, reflecting changing preferences in urban architecture. Fathi and Asgari showed that Iranian-Islamic proportions played a key role in shaping the architecture of the square, and the proportions, which vary depending on the period, reflect not only aesthetic preference but also cultural changes over time.

The geometric patterns adorning the Khoja Akhmet Yasawi Mausoleum share significant similarities with the architectural monuments of Samarkand and Isfahan, particularly in their mathematical foundations and symbolic significance. However,

they also exhibit distinct regional characteristics. The Khoja Akhmet Yasawi Mausoleum's patterns demonstrate mathematical principles derived from Al-Farabi and Al-Khwarizmi's works, particularly in their use of irrational relations (1:√2, 1:√3, 1:√5) and the division of squares into complex geometric forms. This mathematical precision is similarly evident in Samarkand's Registan Square monuments, especially the Sher-Dor Madrasa. However, while the Yasawi Mausoleum emphasizes dark blue brick mosaics with 45-degree angular patterns, Samarkand's monuments typically feature more intricate colour combinations and curved geometric forms. A notable distinction lies in the integration of calligraphy within geometric patterns. The Yasawi Mausoleum incorporates specific phrases into its geometric framework, particularly on its northeastern and southwestern walls. In contrast, Isfahan's Friday Mosque and Samarkand's Registan complex typically separate their calligraphic elements from geometric patterns, creating distinct decorative zones.

The octagonal star patterns found in the Yasawi Mausoleum's central design reflect a particular Central Asian interpretation of Islamic geometry. While similar octagonal formations appear in Isfahan's monuments, they often incorporate more elaborate interlacing patterns and additional geometric layers. Samarkand's architectural patterns demonstrate a greater emphasis on multiple-point star patterns. The use of light and shadow in the designs also differ across these architectural traditions. While octagonal stars exists in both Samarkand and Isfahan monuments, the Yasawi Mausoleum's relatively simple pattern makes its effect more pronounced and intentional. Construction techniques also reveal interesting distinctions. The Yasawi Mausoleum primarily uses burnt bricks of different colours arranged in geometric patterns, with a particular emphasis on dark blue brick mosaics. Isfahan's monuments demonstrate a greater use of glazed tiles and more varied colour palettes, while Samarkand's buildings often combine both techniques with additional decorative elements. The scale and proportion of geometric patterns also vary. The Yasawi Mausoleum's patterns are generally larger and boldly defined. In contrast, Samarkand's and Isfahan's monuments often feature more intricate, smaller-scale patterns that create complex visual tapestries.

The Khoja Akhmet Yasawi Mausoleum, in turn, has undergone minor changes for restoration, through which its original appearance has almost completely been preserved. The geometric patterns of the mausoleum can still be seen on the facades of the building, but not all tiles have been preserved. The researchers described the architecture, exterior, and interior of the Khoja Akhmet Yasawi Mausoleum, and also provided evidence about the patterns of the rosette placed in the centre of the scalloped medallion. Since it has not survived, one can only guess its original pattern. The researchers suggest that the rosette pattern could be based on a grid of a complex geometry, where its design likely applied different geometric laws (Samoilov et al. 2023).

In summation, it can be noted that the geometric patterns in the Khoja Akhmet Yasawi complex stem from laborious work of architects of Medieval Central Asia, who understood geometry well. The patterns use many different geometric laws described by Al-Farabi and Al-Khwarizmi, both respected mathematicians of their time, and reflect their deep mathematical foundations. The methods used are mostly based on various irrational relations, the division squares, equilateral polygons, and triangles.

Based on these geometric methods, the facade of Khoja Akhmet Yasawi achieved its distinctive dynamic features. Throughout the day, as the sun's rays move along the facade, it feels as if the pattern moves. Ornaments do not only have an aesthetic function but also play a symbolic role in relation to Islam. Various words and phrases related to the Muslim religion, preserved since the construction of the complex, can be found on the side facades of the mausoleum in the spaces between the patterns. The doctrines of symmetry and harmony played an important role in the design of the geometric patterns of building facades because they show symmetry and balance, increasing from the main idea up to the level of the ornament.

Conclusion

The Mausoleum of Khoja Akhmet Yasawi is an example of medieval Islamic architecture, combining art, spirituality, and mathematical precision. This research highlighted the multifaceted significance of the mausoleum, illustrating how it embodies the rich cultural heritage of Central Asia and reflects the advanced geometric principles established by Al-Farabi and Al-Khwarizmi. The study underscores the essential role that mathematics played in shaping the design and aesthetics of the mausoleum.

During the study, the first comprehensive analysis linking Al-Farabi and Al-Khwarizmi's mathematical theories to practical architectural implementation in the Khoja Akhmet Yasawi mausoleum was conducted. The article unveiled how mathematical principles were transformed into architectural design during the Timurid period, giving insight into the development of Islamic architectural geometry in Central Asia.

One of the key findings is the systematic application of geometric principles in the construction of the mausoleum. The intricate patterns and motifs on its facades are not merely decorative; they are deeply rooted in mathematical concepts. The works of Al-Khwarizmi, particularly his contributions to algebra and geometry, provided a framework that guided the architects. The use of proportions, symmetry, and repeating elements reflects a profound understanding of mathematical relationships. Furthermore, the research emphasized the importance of situating the mausoleum within its historical and cultural context. Built during the 14th century under the patronage of Amir Temir, the mausoleum symbolizes the sociopolitical and religious aspirations of the time.

Another significant contribution of this research is its exploration of the heritage conservation of the mausoleum, which faces many challenges relating to preservation and restoration. The ongoing debates regarding whether to restore the structure to its original state or to maintain its current condition underscore the complexities of heritage conservation. This study advocates for a balanced approach that respects the historical integrity of the monument while ensuring its relevance to contemporary audiences.

The research results are limited due to the following factors: some patterns and tiles are not preserved in their original state; the inability to verify exact construction methods used by the original builders; the limited ability to reconstruct original design intentions; the difficulty in determining exact mathematical calculations used

during construction; the limited historical documentation of the construction process. Future studies should focus on geometric patterns and other architectural elements, exploring not only the external facade patterns but also the interiors as well.

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Data Availability The data that support the findings of this study are available on request from the corresponding author.

Declarations

Conflict of interest The authors declare that there is no conflict of interests.

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