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Shifted Legendre method with residual error estimation for delay linear Fredholm integro-differential equations

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Abstract

In this paper, we suggest a matrix method for obtaining the approximate solutions of the delay linear Fredholm integro-differential equations with constant coefficients using the shifted Legendre polynomials. The problem is considered with mixed conditions. Using the required matrix operations, the delay linear Fredholm integro-differential equation is transformed into a matrix equation. Additionally, error analysis for the method is presented using the residual function. Illustrative examples are given to demonstrate the efficiency of the method. The results obtained in this study are compared with the known results.
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Keywords: Delay Fredholm integro-differential equations; Numerical solution; Matrix method; Shifted Legendre polynomials

1. Introduction

Delay differential equations and delay integro-differential equations have major importance in many applied areas, including engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, and electrostatics. Therefore, in this paper, we present a new matrix method for solutions of the delay linear Fredholm integro-differential equations with constant coefficients in the form

$$\sum_{k=0}^{m} F_k y^{(k)}(x+\tau_k) = g(x) + \int_0^1 \sum_{s=0}^{m} K_s(x,t) y^{(s)}(t+\gamma_s)dt, \quad 0 \le x, t \le 1 \tag{1}$$

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under mixed conditions

$$\sum_{k=0}^{m-1}(a_{jk}y^{(k)}(0)+b_{jk}y^{(k)}(1))=\mu_j,\quad j=0,1,\ldots,m-1 \tag{2}$$

where F_k, a_{jk}, b_{jk}, τ_k, γ_s and μ_j are real constants; $y^{(0)}(x)=y(x)$ is an unknown function; $g(x)$ and $K_s(x,t)$ are the functions defined on the interval $0\le x,\ t\le 1$; and $g(x)$ and $K_s(x,t)$ can be represented by Maclaurin series.

Our purpose in this study is to determine the approximate solutions of Eq. (1) under conditions (2) in the form

$$y_N(x)=\sum_{n=0}^{N}a_nL_n(x). \tag{3}$$

In this equation, a_n $(n=0,1,2,\ldots,N)$ are the unknown Legendre coefficients and $L_n(x)$ $(n=0,1,2,\ldots)$ denote the shifted Legendre polynomials, which are defined by (Rodriguez formula)

$$L_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n,\quad n\in\mathbb{N};\quad 0\le x\le 1.$$

In recent years, many authors have studied numerical methods for the approximate solutions of integro-differential equations [1–14].

2. Fundamental matrix relations

Let us show Eq. (1) in the form

$$D(x)=g(x)+I(x) \tag{4}$$

where

$$D(x)=\sum_{k=0}^{m}F_ky^{(k)}(x+\tau_k) \tag{5}$$

and

$$I(x)=\int_0^1\sum_{s=0}^{m}K_s(x,t)y^{(s)}(t+\gamma_s)dt \tag{6}$$

Now we convert the solution $y(x)$ and its kth order derivative $y^{(k)}(x)$, the parts $D(x)$ and $I(x)$, and the mixed conditions (2) to the matrix forms.

2.1. Matrix relation for the differential part D(x)

First, we write the approximate solution $y(x)$ of Eq. (1) defined by the truncated Legendre series (3) in the matrix form

$$y(x)=\mathbf{L}(x)\mathbf{A} \tag{7}$$

where

$$\mathbf{L}(x)=[\,L_0(x)\quad L_1(x)\quad\ldots\quad L_N(x)\,],\quad \mathbf{A}=[\,a_0\quad a_1\quad\ldots\quad a_N\,]^{\mathrm{T}}.$$

The matrix form $\mathbf{L}(x)$ can be written as

$$\mathbf{L}(x)=\mathbf{X}(x)\mathbf{Ş}\quad\text{or}\quad\mathbf{X}(x)=\mathbf{L}(x)\mathbf{Ş}^{-1} \tag{8}$$

where

$$\mathbf{X}(x)=\begin{bmatrix}1 & x & x^2 & \ldots & x^N\end{bmatrix},\quad \mathbf{Ş}=\mathbf{M}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}},$$

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N!} \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} (-1)^0 \binom{0}{0} \frac{0!}{0!} & 0 & 0 & \cdots & 0 \\ (-1)^1 \binom{1}{1} \frac{1!}{0!} & (-1)^0 \binom{1}{0} \frac{2!}{1!} & 0 & \cdots & 0 \\ (-1)^2 \binom{2}{2} \frac{2!}{0!} & (-1)^1 \binom{2}{1} \frac{3!}{1!} & (-1)^0 \binom{2}{0} \frac{4!}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^N \binom{N}{N} \frac{N!}{0!} & (-1)^{N-1} \binom{N}{N-1} \frac{(N+1)!}{1!} & (-1)^{N-2} \binom{N}{N-2} \frac{(N+2)!}{2!} & \cdots & (-1)^0 \binom{N}{0} \frac{(2N)!}{N!} \end{bmatrix}.$$

Additionally, the relation between the matrix $\mathbf{X}(x)$ and its kth order derivative $\mathbf{X}^{(k)}(x)$

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{C}^k, \quad k = 0, 1, \ldots, m. \tag{9}$$

where $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$ and $\mathbf{C}^0$ is a unit matrix with dimension $(N+1) \times (N+1)$.

From Eqs. (8) and (9), we obtain the matrix relation

$$\mathbf{L}^{(k)}(x) = \mathbf{L}(x)\mathbf{Ş}^{-1}\mathbf{C}^k\mathbf{Ş}. \tag{10}$$

By placing Eq. (10) into the kth order derivative of Eq. (7), we have the recurrence relation

$$y^{(k)}(x) = \mathbf{L}(x)\mathbf{Ş}^{-1}\mathbf{C}^k\mathbf{Ş}\,\mathbf{A}, \quad k = 0, 1, \ldots, m. \tag{11}$$

By putting $x \to x + \tau_k$ in the relation (8), we obtain the matrix relation

$$\mathbf{L}(x + \tau_k) = \mathbf{X}(x + \tau_k)\mathbf{Ş} = \mathbf{X}(x)\mathbf{B}(\tau_k)\mathbf{Ş} \tag{12}$$

where

$$\mathbf{B}(\tau_k) = \begin{bmatrix} \binom{0}{0}(\tau_k)^0 & \binom{1}{0}(\tau_k)^1 & \cdots & \binom{N}{0}(\tau_k)^N \\ 0 & \binom{1}{1}(\tau_k)^0 & \cdots & \binom{N}{1}(\tau_k)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\tau_k)^0 \end{bmatrix}.$$

From Eqs. (8) and (12), we have

$$\mathbf{L}(x+\tau_k) = \mathbf{L}(x)\mathbf{Ş}^{-1}\mathbf{B}(\tau_k)\mathbf{Ş}. \tag{13}$$

By placing $x \to x+\tau_k$ in the relation (11) and by using Eq. (13), we obtain the recurrence relation

$$y^{(k)}(x+\tau_k) = \mathbf{L}(x)\mathbf{Ş}^{-1}\mathbf{B}(\tau_k)\mathbf{C}^k\mathbf{Ş}\,\mathbf{A}. \tag{14}$$

By substituting the expression (14) into (5) we have the matrix form

$$D(x) = \sum_{k=0}^{m} F_k\mathbf{L}(x)\mathbf{Ş}^{-1}\mathbf{B}(\tau_k)\mathbf{C}^k\mathbf{Ş}\,\mathbf{A}. \tag{15}$$

2.2. Matrix relation for the integral part I(x)

Because we assume that the kernel functions $K_s(x,t)$ for $s=0,1,\ldots,m$ can be expanded to Maclaurin series, the kernel functions $K_j(x,t)$ for $s=0,1,\ldots,m$ can be approximated by a truncated Maclaurin series in the form

$$K_s(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}^{T}k_{mn}^{s}x^m t^n \tag{16}$$

where

$${}^{T}k_{mn}^{s} = \frac{1}{m!n!}\frac{\partial^{m+n}K_s(0,0)}{\partial x^m\partial t^n}; \quad m,n=0,1,2,\ldots,N, \quad s=0,1,\ldots,m.$$

Eq. (16) can be expressed in the matrix form

$$K_s(x,t) = \mathbf{X}(x)\mathbf{K}_T^s\mathbf{X}^T(t) \tag{17}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix},$$
$$\mathbf{K}_T^s = \begin{bmatrix} {}^{T}k_{mn}^{s} \end{bmatrix}, \quad m,n=0,1,2,\ldots,N \quad \text{and} \quad j=0,1,\ldots,J.$$

Additionally, the kernel function $K_s(x,t)$ can be approximated by the truncated shifted Legendre series of degree N

$$K_s(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}^{L}k_{mn}^{s}L_m(x)L_n(t), \quad s=0,1,\ldots,m$$

and the matrix form based on the shifted Legendre series of the function $K_s(x,t)$ becomes

$$K_s(x,t) = \mathbf{L}(x)\mathbf{K}_L^s\mathbf{L}^T(t); \quad \mathbf{K}_L^s = \begin{bmatrix} {}^{L}k_{mn}^{s} \end{bmatrix}. \tag{18}$$

Now, let us try to find a relation between the matrices $\mathbf{K}_L^s$ and $\mathbf{K}_T^s$. First, we write the following relation from Eq. (8):

$$\mathbf{X}(x) = \mathbf{L}(x)\mathbf{Ş}^{-1}. \tag{19}$$

By equating (17) and (18) and by using Eq. (19), we obtain the relation

$$\mathbf{K}_L^s = \mathbf{Ş}^{-1}\mathbf{K}_T^s(\mathbf{Ş}^{-1})^{\mathrm{T}}. \tag{20}$$

By placing $x+\tau_k \to t+\gamma_s$ in the relation (14), we have the relation

$$y^{(k)}(t+\gamma_s) = \mathbf{L}(t)\mathbf{Ş}^{-1}\mathbf{B}(\gamma_s)\mathbf{C}^k\mathbf{Ş}\,\mathbf{A}. \tag{21}$$

By substituting the matrix forms (18) and (21) into Eq. (6) we have the matrix relation

$$\mathbf{I}(x) = \int_0^1 \sum_{s=0}^{m} \mathbf{L}(x)\mathbf{K}_L^s\mathbf{L}^T(t)\mathbf{L}(t)\mathbf{Ş}^{-1}\mathbf{B}(\gamma_s)\mathbf{C}^s\mathbf{Ş}\,\mathbf{A}\,dt = \mathbf{L}(x)\left\{\sum_{s=0}^{m}\mathbf{K}_L^s\mathbf{Q}\mathbf{Ş}^{-1}\mathbf{B}(\gamma_s)\right\}\mathbf{C}^s\mathbf{Ş}\,\mathbf{A} \tag{22}$$

where

$$\mathbf{Q} = \int_0^1 \mathbf{L}^{\mathrm{T}}(t)\mathbf{L}(t)dt = [q_{ls}]; \quad l,s = 0,1,\ldots,N.$$

From Eq. (8), the matrix $\mathbf{Q}$ is briefly computed with the matrix form

$$\mathbf{Q} = \mathbf{Ş}^{\mathrm{T}}\mathbf{H}\mathbf{Ş}$$

so that

$$\mathbf{H} = \int_0^1 \mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)dt = [h_{k,r}]; \quad h_{k,r} = \frac{1}{k+r+1}, \quad k,r = 0,1,\ldots,N.$$

2.3. Matrix relation for the initial-boundary conditions

From Eq. (11), we obtain the corresponding matrix forms for the conditions (2) as

$$\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{L}(0) + b_{jk}\mathbf{L}(1)\right]\mathbf{Ş}^{-1}\mathbf{C}^k\mathbf{Ş}\,\mathbf{A} = [\mu_j], \quad j = 0,1,\ldots,m-1. \tag{23}$$

2.4. Matrix representation of the function g(x)

The term $g(x)$ of Eq. (1) can be approximated by the truncated Maclaurin series in the form

$$g(x) = \sum_{n=0}^{N} g_n^T x^n; \quad g_n^T = \frac{g^{(n)}(0)}{n!}$$

and the matrix form of the Maclaurin expansion of $g(x)$ becomes

$$g(x) = \mathbf{X}(x)\mathbf{G}_T; \quad \mathbf{G}_T = \begin{bmatrix} g_0^T & g_1^T & \ldots & g_N^T \end{bmatrix}^{\mathrm{T}}. \tag{24}$$

Additionally, $g(x)$ can be approximated by the truncated shifted Legendre series in the form

$$g(x) = \sum_{n=0}^{N} g_n^L L_n(x) = \mathbf{L}(x)\mathbf{G}_L, \quad \mathbf{G}_L = \begin{bmatrix} g_0^L & g_1^L & \ldots & g_N^L \end{bmatrix}^{\mathrm{T}}. \tag{25}$$

By equating (24) and (25) and by using Eq. (19), we have the relation

$$\mathbf{G}_L = \mathbf{Ş}^{-1}\mathbf{G}_T$$

and thus, the shifted Legendre expansion of $g(x)$ is computed with

$$g(x) = \mathbf{L}(x)\mathbf{G}_L \tag{26}$$

so that $\mathbf{G}_L = \mathbf{Ş}^{-1}\mathbf{G}_T$.

3. Method of solution

We are now ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, substituting the matrix relations (15), (22) and (26) into Eq. (4) and simplifying, we obtain the fundamental matrix equation

$$\left\{\sum_{k=0}^{m} F_k \mathbf{S}^{-1}\mathbf{B}(\tau_k)\mathbf{C}^k\mathbf{S} - \sum_{s=0}^{m}\mathbf{K}_L^s\mathbf{Q}\mathbf{S}^{-1}\mathbf{B}(\gamma_s)\mathbf{C}^s\mathbf{S}\right\}\mathbf{A} = \mathbf{G}_L. \tag{27}$$

Eq. (27) corresponds to a system of $(N+1)$ algebraic equations for the unknown coefficients $a_0, a_1, \ldots, a_n$. Briefly, we can write Eq. (27) in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G}_L \quad \text{or} \quad [\mathbf{W}; \mathbf{G}_L] \tag{28}$$

where

$$\mathbf{W} = [w_{p,q}] = \sum_{k=0}^{m} F_k \mathbf{S}^{-1}\mathbf{B}(\tau_k)\mathbf{C}^k\mathbf{S} - \sum_{s=0}^{m}\mathbf{K}_L^s\mathbf{Q}\mathbf{S}^{-1}\mathbf{B}(\gamma_s)\mathbf{C}^s\mathbf{S}\,; \quad p, q = 0, 1, \ldots, N.$$

Conversely, the matrix form (23) for the conditions (2) can be written as

$$\mathrm{U}_i\mathrm{A} = [\mu_i] \quad \text{or} \quad [\mathrm{U}_i; \mu_i]; \quad i = 0, 1, 2, \ldots, m-1 \tag{29}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1}\left[a_{jk}\mathbf{L}(0) + b_{jk}\mathbf{L}(1)\right]\mathbf{S}^{-1}\mathbf{C}^k\mathbf{S}\,\mathbf{A} = [u_{i,0}, u_{i,1}, \ldots, u_{i,N}], \quad i = 0, 1, \ldots, m-1.$$

To obtain the solution of Eq. (1) under the conditions (2), by replacing the m row matrices (29) by the m rows of the matrix (28) we have the new augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}} \quad \text{or} \quad [\tilde{\mathbf{W}}; \tilde{\mathbf{G}}].$$

If $\operatorname{rank}\tilde{\mathbf{W}} = \operatorname{rank}[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = N+1$, the unknown coefficients matrix $\mathbf{A}$ becomes

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}.$$

Hence, the Legendre coefficients matrix $\mathbf{A}$ is uniquely determined. By substituting the determined coefficients $a_0, a_1, \ldots, a_n$ into Eq. (7), we obtain the Legendre polynomial solution

$$y_N(x) = \sum_{n=0}^{N} a_n L_n(x). \tag{30}$$

4. Error estimation

In this section, an error estimation based on a residual function [14–16] is given for the Legendre approximate solution of Eq. (1). This error estimation was presented for the Bessel approximate solutions of a system of linear multi-pantograph equations in [17]. For the linear delay Fredholm integro-differential equations, we modify the error estimation studied in [14–17].

Let us define $e_N(x) = y(x) - y_N(x)$ as the error function of the Legendre approximation $y_N(x)$ to $y(x)$, where $y(x)$ is the exact solution of the problems (1) and (2). Hence, $y_N(x)$ satisfies the following problem:

$$\sum_{k=0}^{m} F_k y_N^{(k)}(x+\tau_k) - \int_0^1 \sum_{s=0}^{m} K_s(x,t) y_N^{(s)}(t+\gamma_s)dt = g(x) + R_N(x), \quad x \in [0,1] \tag{31}$$

$$\sum_{k=0}^{m-1}\left(a_{jk}y_N^{(k)}(0)+b_{jk}y_N^{(k)}(1)\right)=\mu_j,\quad j=0,1,\ldots,m-1. \tag{32}$$

which can be obtained by substituting $y_N(x)$ into Eq. (1) and in here $R_N(x)$ is the residual function associated with $y_N(x)$.

By means of the method defined for the problems (1) and (2) in Section 3, we proceed to find an approximation $e_{N,M}(x)$ to the $e_N(x)$.

By subtracting (31) and (32) from (1) and (2), respectively, the error function $e_N(x)$ satisfies the equation

$$\sum_{k=0}^{m}F_k e_N^{(k)}(x+\tau_k)-\int_0^1\sum_{s=0}^{m}K_s(x,t)e_N^{(s)}(t+\gamma_s)dt=-R_N(x),\quad x\in[0,1] \tag{33}$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1}\left(a_{jk}e_N^{(k)}(0)+b_{jk}e_N^{(k)}(1)\right)=0,\quad j=0,1,\ldots,m-1. \tag{34}$$

By solving the error problems (33) and (34) in the same way as in Section 3, the approximation $e_{N,M}(x)$ to $e_N(x)$ is obtained. Hence, if the exact solution of the problem is not known, then we compute the error function, approximately.

5. Numerical examples

In this section, the efficiency of the method is demonstrated with the numerical results of two examples. In tables and figures, we give the values of the exact solution $y(x)$, the approximate solution $y_N(x)$, the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error function $|e_{N,M}(x)|$ at the selected points of the given interval. All examples were solved by a computer code written in MATLAB.

Example 1. Let us first consider the linear delay Fredholm integro-differential equation

$$y^{(2)}(x+1)+2y^{(1)}(x-0.5)-y(x-0.2)=g(x)+\int_0^1\left[\cos(x)ty(t+1)-\sin(x)ty^{(1)}(x+0.5)\right]dt \tag{35}$$

with the initial conditions $y(0)=1$ and $y'(0)=0$ and the exact solution $y(x)=\cos(x)$.

Here, $g(x)=-\cos(x+1)-2\sin(x-0.5)-\cos(x-0.2)+\cos(x)\cos(1)+\sin(x)\sin(0.5)-\cos(x)\cos(2)$, $-\cos(x)\sin(2)-\sin(x)\sin(1.5)+\sin(x)\cos(1.5)$, $F_0=-1$, $F_1=2$, $F_2=1$, $m=2$, $K_0(x,t)=\cos(x)t$ and $K_1(x,t)=-\sin(x)t$.

By applying the method presented for $N=5$, 10, we obtain the approximate solutions

$$\begin{aligned}y_5(x)=\;&1+(0.328716546170e-16)x-(0.459471186737)x^2+(0.255086330744e-1)x^3\\&+(0.395058723740e-1)x^4-(0.601050087863e-2)x^5,\end{aligned}$$

$$\begin{aligned}y_{10}(x)=\;&+1+(0.107485483688e-16)x-(0.512747818208)x^2-(0.754277666517e-2)x^3\\&+(0.428447538578e-1)x^4+(0.999418470044e-3)x^5-(0.141081236455e-2)x^6\\&-(0.523927330760e-4)x^7+(0.235123390721e-4)x^8+(0.163065575297e-5)x^9\\&-(0.323339175181e-6)x^{10}.\end{aligned}$$

By using the error estimation presented in Section 4, the error functions for the above approximate solutions are estimated. For some values of N and M, the actual absolute error functions are compared with the estimated absolute error functions in Fig. 1(a)–(c). In Fig. 1(d), we compare the estimated absolute error functions for different values of (N,M). It is seen from Fig. 1 that the errors decrease as values N and M increase.

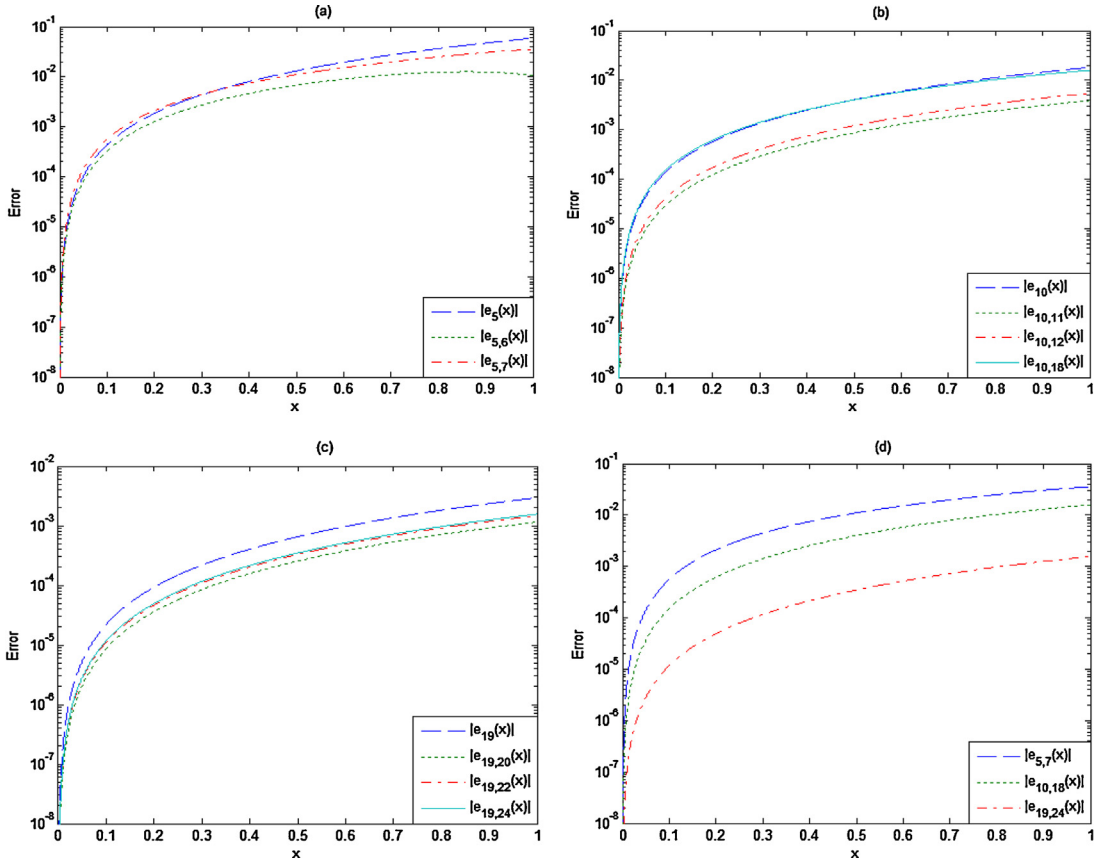

Fig. 1. (a) and (b) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for some values of Eq. (35). (c) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for $N = 19$ and $M = 20, 22, 24$ of Eq. (35). (d) Comparison of the estimated absolute error functions $|e_{N,M}(x)|$ for $(N,M) = (5,7)$, (10,18), (19,24) of Eq. (35).

Example 2. [7] Now we consider the Firedholm integro-differential equation

$$y'(x) - y(x) = +\frac{1 - e^{x+1}}{x+1} + \int_0^1 e^{tx} y(t)dt \tag{36}$$

with the initial condition $y(0) = 1$. The exact solution of the problem is given by $y(x) = e^x$.

The absolute errors are compared with the homotopy perturbation method (HPM) [7], the differential transformation method (DTM) [12] and the CAS wavelet method (CASWM) [13] in Table 1.

Table 1
Comparison of the absolute errors of Eq. (36).

x_i	HPM [7]	DTM [12]	CASWM [13]	Present method		
				$e_5(x_i)$	$e_8(x_i)$	$e_{12}(x_i)$
0.1	2.314814815e−06	1.00118319e−02	1.34917637e−03	2.4153e−007	1.4354e−009	6.8889e−014
0.2	9.259259259e−06	2.78651355e−02	1.15960044e−03	1.0012e−005	6.8021e−009	2.7522e−013
0.4	3.703703704e−05	7.55356316e−02	5.93105645e−02	1.8641e−005	2.6481e−008	1.1017e−012
0.6	8.333333333e−05	1.09551714e−01	4.39287720e−02	8.2518e−005	7.7519e−008	2.6095e−012
0.8	1.481481481e−04	6.94512700e−02	1.34514117e−02	4.0963e−004	4.2704e−007	1.1894e−011
0.9	1.875000000e−04	1.00034260e−02	1.32045209e−02	8.5537e−004	1.1891e−006	4.4825e−011

6. Conclusions

In this paper, we presented a matrix scheme based on the shifted Legendre polynomials for the delay linear Fredholm integro-differential equations with constant coefficients. Additionally, we presented an error estimation method in terms of the residual function. It is seen from Example 1 that error estimation is highly effective. If the exact solution of the problem is unknown, then the absolute errors can be approximately computed with this error estimation. The comparison of the suggested method with the other methods shows that our method is highly effective. A considerable advantage of the method is that the approximate solutions are computed very easily using a well-known symbolic software such as MATLAB, Maple or Mathematica.

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