Parallel Frames of Non-Lightlike Curves

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Abstract

The parallel frame of a curve is an alternative approach to defining a moving frame. In this paper, we express the parallel frames of timelike and spacelike curves in Minkowski space.

Keywords : Minkowski space, Parallel frame, Bishop Frame, Timelike curve, Spacelike curve.

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1 Introduction

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. Curvature may be vanish at some points for a curve. That is, second derivative of a curve may be zero. In this situation, we can use an alternative frame. It is used tangent vector $\mathbf{T}$ and two relatively parallel vector fields to construct this alternative frame such that a normal vector field $\mathbf{N}$ along a curve is relatively parallel if its derivative is tangential. We call this frame parallel frame along $\mathbf{T}$. The reason for the name parallel is because the normal component of the derivatives of the normal vector fields is zero. The advantages of the parallel frame and the comparable parallel frame with the Frenet frame in Euclidean 3-space was given and studied by [1] and Hanson ([3], [4]).

Basic concept of this frame is taking unique tangent vector and choosing any convenient basis $\{\mathbf{N}_1(s), \mathbf{N}_2(s)\}$ in the plane perpendicular to $\mathbf{T}(s)$ at each point such that derivatives of $\{\mathbf{N}_1(s), \mathbf{N}_2(s)\}$ depend only $\mathbf{T}(s)$ and not each other. Parallel frame is not unique contrast to Frenet frame.

In the three dimensional Euclidean space the alternative parallel frame equations are

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{bmatrix}$$

for a parametrized unit length curve. The relation between Frenet frame and parallel frame as follows:

$$\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{bmatrix}$$

such that $\sin\theta = \dfrac{k_1}{\kappa}$ and $\cos\theta = \dfrac{k_2}{\kappa}$.Also, $\kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}$, $\theta(s) = \arctan\dfrac{k_2}{k_1}$ and $\tau(s) = \theta'(s)$ so that $k_1(s)$ and $k_2(s)$ correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)\, ds$.

The functions $k_1(s)$, $k_2(s)$ are called the principal curvature along $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$ respectively. Also, $(k_1(s), k_2(s))$ can be seen a sort of invariant of the curve. This is more difficult to conceive than in the case (κ, τ), since parallel frame is not unique. We call $(k_1(s), k_2(s))$ the normal development of the curve. If two regular curves in Euclidean space have the same normal development, then curves are congruent [1].

2 Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle \mathbf{u}, \mathbf{v} \rangle_L = -u_1v_1 + u_2v_2 + u_1v_2$ where $\mathbf{u} = (u_1, u_1, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\mathbf{u}$ in E_1^3 is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle_L > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_L = 0$ or $\langle \mathbf{u}, \mathbf{u} \rangle_L < 0$ respectively. The norm of the vector $\mathbf{u} \in \mathbb{E}_1^3$ is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle_L|}$. For any $\mathbf{u} = (u_1, u_1, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{E}_1^3$, the Lorentzian vector product $\mathbf{u} \times_L \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined as follows:

$$\mathbf{u} \times_L \mathbf{v} = (-u_2v_3 + u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

An arbitrary curve $\alpha = \alpha(s) : I \to \mathbb{E}_1^3$ is spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset R$. Throughout this paper we shall assume all curves are parametrized by its arc length.

Let $\alpha(s)$ be a non-lightlike curve and $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ (\epsilon_{\mathbf{B}})\kappa(s) & 0 & \tau(s) \\ 0 & (\epsilon_{\mathbf{T}})\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} \tag{1}$$

where $\epsilon_{\mathbf{X}} = \langle \mathbf{X}, \mathbf{X} \rangle_L$ and $\kappa(s)$, $\tau(s)$ are curvature function and torsion function respectively [5], [6].

3 Parallel frame of a non-lightlike curve

In this section, we describe parallel frames for timelike and spacelike curves in the Minkowski 3-space using the similar methods in the Euclidean 3-space. Parallel frame is an alternative frame for curves and can be more useful comparing to Frenet frame. Firstly, we state following known lemma.

Proposition 1 *Let $e_1, e_2, ..., e_n : [a, b] \to \mathbb{E}_1^n$ be smooth maps and $\{e_1(s), e_2(s), ..., e_n(s)\}$ be an orthonormal basis of $\mathbb{E}_1^n$ for all $s \in [a, b]$. Then, there exist a matrix valued function $A(s) = (a_{ij}(s))$ so that*

i) $e_i'(s) = \sum_{j=1}^{n} a_{ji}(s) e_j(s)$

ii) $a_{ij} = \epsilon_{e_i} \left\langle e_j', e_i \right\rangle_L$

iii) $\epsilon_{ej} a_{ij} + \epsilon_{e_i} a_{ji} = 0$ *(that is $A(t)$ is semi skew symmetric)*

where $\epsilon_{e_i} = \langle e_i, e_i \rangle_L$.

3.1 Timelike Curves

Let $\{\mathbf{T}, \mathbf{Z}_1, \mathbf{Z}_2\}$ be a smooth orthonormal frame along a unit speed timelike curve $\alpha(s) : I \to \mathbb{E}_1^3$. According to proposition above, there exist smooth functions f_1, f_2 and f_3 such that

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & 0 & f_3 \\ f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix}.$$

Now, let's change the orthonormal frame $\{\mathbf{T}, \mathbf{Z}_1, \mathbf{Z}_2\}$ to $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ such that 23 and 32 entries of obtained matrix of $\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}$ is zero. So, we rotate the spacelike normal vectors $\{\mathbf{Z}_1, \mathbf{Z}_2\}$ by angle $\theta(s)$. That is,

$$\begin{aligned} \mathbf{N}_1 &= (\cos\theta(s))\,\mathbf{Z}_1 - (\sin\theta(s))\,\mathbf{Z}_2 \\ \mathbf{N}_2 &= (\sin\theta(s))\,\mathbf{Z}_1 + (\cos\theta(s))\,\mathbf{Z}_2 \end{aligned}$$

We want that normal component of the derivative of $\mathbf{N}_1$ and $\mathbf{N}_2$ is zero. Namely, $\langle \mathbf{N}_1', \mathbf{N}_2 \rangle_L = 0$ and $\langle \mathbf{N}_1, \mathbf{N}_2' \rangle_L = 0$. So,

$$\begin{aligned} \langle \mathbf{N}_1', \mathbf{N}_2 \rangle_L &= \langle -\theta'(\sin\theta)\,\mathbf{Z}_1 - \theta'(\cos\theta)\,\mathbf{Z}_2 + (\cos\theta(s))\,\mathbf{Z}_1 - (\sin\theta(s))\,\mathbf{Z}_2, \mathbf{N}_2 \rangle_L \\ &= -\theta'\left(\sin^2\theta + \cos^2\theta\right) + \cos\theta\,(f_3\cos\theta) + \sin\theta\,(f_3\sin\theta) \\ &= -\theta' + f_3 \end{aligned}$$

Thus, if we choose $\theta(s)$ as satisfying the equality

$$\theta'(s) = f_3, \tag{2}$$

we obtain a new orthonormal frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ satisfying the derivative formulas

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}.$$

such that $k_1 = \langle \mathbf{T}', \mathbf{N}_1 \rangle_L$ and $k_2 = \langle \mathbf{T}', \mathbf{N}_2 \rangle_L$. Observe that the parallel frame is not unique, since the choose of it depend only (2).

3.2 Spacelike Curves

Now, let's construct parallel frame using the similar way. But, in this case we must consider that one of the $\mathbf{N}_1$ or $\mathbf{N}_2$ is timelike. At first, suppose that $\{\mathbf{T}, \mathbf{Z}_1, \mathbf{Z}_2\}$ be a smooth orthonormal frame along a unit speed spacelike curve $\alpha(s) : I \to \mathbb{E}_1^3$ such that $\mathbf{Z}_1$ is timelike. Then there exist smooth functions f_1, f_2 and f_3 satisfying

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & 0 & f_3 \\ -f_2 & f_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix}$$

where the matrix in the right side is semi skew symmetric with respect to the signature $(+,-,+)$, that is in the order of the causal characters of $\{\mathbf{T}, \mathbf{Z}_1, \mathbf{Z}_2\}$.

To vanish the 23 and 32 entries of the new obtained matrix of $\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}$ in the revolving $\{\mathbf{T}, \mathbf{Z}_1, \mathbf{Z}_2\}$ to $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$. So, we rotate the normal vectors $\{\mathbf{Z}_1, \mathbf{Z}_2\}$ by angle $\theta(s)$. But, in this case, we express rotation matrix with hyperbolic functions since $\mathbf{Z}_1$ is timelike. For the rotation matrices in Minkowski space, we refer to [7]. So we can write as

$$\mathbf{N}_1 = (\cosh\theta(s))\,\mathbf{Z}_1 + (\sinh\theta(s))\,\mathbf{Z}_2$$
$$\mathbf{N}_2 = (\sinh\theta(s))\,\mathbf{Z}_1 + (\cosh\theta(s))\,\mathbf{Z}_2.$$

Similarly, we want to vanish the normal component of the derivative of $\mathbf{N}_1$ and $\mathbf{N}_2$. Therefore, the equalities

$$\langle \mathbf{N}_1', \mathbf{N}_2\rangle_L = 0 \text{ and } \langle \mathbf{N}_1, \mathbf{N}_2'\rangle_L = 0$$

must satisfy. Thus, considering also the normal vector M_1 is timelike,

$$\begin{aligned}\langle \mathbf{N}_1', \mathbf{N}_2\rangle_L &= \langle \theta'(\sinh\theta)\,\mathbf{Z}_1 + \theta'(\cosh\theta)\,\mathbf{Z}_2 + (\cosh\theta(s))\,\mathbf{Z}_1 + (\sinh\theta(s))\,\mathbf{Z}_2, \mathbf{N}_2\rangle_L \\ &= \theta'\left(-\sinh^2\theta + \cosh^2\theta\right) + (\cosh\theta)\,(f_3\cosh\theta) - \sinh\theta\,(f_3\sinh\theta) \\ &= \theta' + f_3\end{aligned}$$

Thus, if we choose $\theta(s)$ as satisfying the equality

$$\theta'(s) = -f_3, \tag{3}$$

we obtain a new orthonormal frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$satisfying the derivative formulas

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}.$$

such that $k_1 = -\langle \mathbf{T}', \mathbf{N}_1\rangle_L$ and $k_2 = \langle \mathbf{T}', \mathbf{N}_2\rangle_L$.

For the third case, that is, for the spacelike curves such that $\mathbf{Z}_2$ is timelike, we find

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}$$

such that $k_1 = \langle \mathbf{T}', \mathbf{N}_1\rangle_L$ and $k_2 = -\langle \mathbf{T}', \mathbf{N}_2\rangle_L$.

So, we describe parallel frame of the spacelike and timelike curves in Minkowski 3 space. We can summarize derivative formulas of parallel frame for timelike and spacelike curves as follows.

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -\epsilon_{\mathbf{N}_1} k_1 & 0 & 0 \\ -\epsilon_{\mathbf{N}_2} k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix} \tag{4}$$

where $k_1 = \epsilon_{\mathbf{N}_1}\langle \mathbf{T}', \mathbf{N}_1\rangle_L$, $k_2 = \epsilon_{\mathbf{N}_1}\langle \mathbf{T}', \mathbf{N}_2\rangle_L$ with $\epsilon_{\mathbf{X}} = \langle \mathbf{X}, \mathbf{X}\rangle_L$. The functions k_1 and k_2 are called the principal curvatures along $\mathbf{N}_1$ and $\mathbf{N}_2$. Also, the parallel frame is not unique, since the choose of the parallel frame depend only the (2) and (3).

Also, we assume that $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ is positively oriented and vector products of these vectors defined as follows:

$$\mathbf{T} \times_L \mathbf{N}_1 = \epsilon_{\mathbf{N}_2}\mathbf{N}_2,\ \mathbf{N}_1 \times_L \mathbf{N}_2 = \epsilon_{\mathbf{T}}\mathbf{T} \text{ and } \mathbf{N}_2 \times_L \mathbf{T} = \epsilon_{\mathbf{N}_1}\mathbf{N}_1.$$

In the Minkowski 3-space, the Darboux vector $\mathbf{w}$ of Frenet frame of a curve α as follows:
1) If α is timelike, $\mathbf{w}_f = \tau\mathbf{T} + \kappa\mathbf{B}$,
2) If α is spacelike with timelike normal, $\mathbf{w}_f = \tau\mathbf{T} - \kappa\mathbf{B}$,
3) If α is spacelike with timelike binormal, $\mathbf{w}_f = -\tau\mathbf{T} + \kappa\mathbf{B}$.

For parallel frame of a timelike curve, considering the parallel frame derivative formulas and properties of the vector products of $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$. it can be seen that the equalities

$$\frac{d\mathbf{T}}{ds} = \mathbf{w}_p \times_L \mathbf{T},\ \frac{d\mathbf{N}_1}{ds} = \mathbf{w}_p \times_L \mathbf{N}_1,\ \frac{d\mathbf{N}_2}{ds} = \mathbf{w}_p \times_L \mathbf{N}_2$$

are satisfied in the case the vector $\mathbf{w}_p = -k_2\mathbf{N}_1 + k_1\mathbf{N}_2$. So,
1) The Darboux vector of the parallel frame of a timelike curve is $\mathbf{w}_p = -k_2\mathbf{N}_1 + k_1\mathbf{N}_2$.
In the similar way, we find the Darboux vector of a spacelike curve in the Minkowski 3-space as follows:
1) If $\mathbf{N}_1$ is timelike, then $\mathbf{w}_p = -k_2\mathbf{N}_1 - k_1\mathbf{N}_2$
2) If $\mathbf{N}_2$ is timelike then $\mathbf{w}_p = k_2\mathbf{N}_1 + k_1\mathbf{N}_2$.

3.3 The relation between Frenet frame and parallel frame of a non-lightlike curve

Let $\alpha(s) : I \to \mathbb{E}_1^3$ be a non-lightlike curve in the Minkowski 3-space. Also, let Frenet frame and parallel frame of this curve be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ respectively.

From the Frenet formulas, we know that $\mathbf{T}' = \kappa\mathbf{N}$. So, using this equality and parallel frame derivative formulas, we find

$$\mathbf{N} = \frac{\mathbf{T}'}{\kappa} = \frac{k_1\mathbf{N}_1 + k_2\mathbf{N}_2}{\kappa}. \tag{5}$$

Taking the vector product of $\mathbf{T}$ from left with both sides of the equation (5), we obtain

$$B = \frac{\epsilon_{\mathbf{N}_2} k_1\mathbf{N}_2 - \epsilon_{\mathbf{N}_1} k_2\mathbf{N}_1}{\epsilon_{\mathbf{B}}\kappa}. \tag{6}$$

Now, we examine the equations (5) and (6), with respect to causal character of the curve.
Case 1: If curve is timelike
In this case, if we write $\cos\theta = \frac{k_1}{\kappa}$ and $\sin\theta = \frac{k_2}{\kappa}$ in (5) and (6), we find the equation

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}.$$

It means that, the parallel frame of the a timelike curve is obtained with revolving the Frenet frame about the timelike vector $\mathbf{T}$ with angle θ.
Case 2: If curve is spacelike with timelike normal $\mathbf{N}_1$.
If we write $\epsilon_{\mathbf{N}_1} = -1$, $\cosh\theta = \frac{k_1}{\kappa}$ and $\sinh\theta = \frac{k_2}{\kappa}$ in the equations (5) and (6), we find the equation

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}.$$

Case 3: If curve is spacelike with timelike normal $\mathbf{N}_2$.
In this case, B is also timelike and if we write again $\epsilon_{\mathbf{N}_2} = \epsilon_{\mathbf{B}} = -1$, $\cosh\theta = \frac{k_1}{\kappa}$ and $\sinh\theta = \frac{k_2}{\kappa}$ in the equations (5) and (6), we obtain

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}.$$

Corollary 2 *The parallel frame of the a non-lightlike curve in Minkowski 3-space is obtained with revolving the Frenet frame of the curve about the speed vector* $\mathbf{T}$ *with angle* $\theta(s) = \arctan\frac{k_2}{k_1}(s)$.

Corollary 3 *Causal characters of the* $\mathbf{N}$ *and* $\mathbf{N}_1$ *or* $\mathbf{B}$ *and* $\mathbf{N}_2$ *are same respectively.*

Lemma 4 *Let α be a unit speed timelike curve then*

$$\kappa(s) = \sqrt{k_1^2 + k_2^2} \quad \text{and} \quad \tau(s) = \theta'(s), \theta(s) = \arctan\frac{k_2}{k_1}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion functions of the curve and k_1, k_2 are principal curvatures along the $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively.

Proof. Using the equation $\kappa = \|T'\|$, we obtain $\kappa(s) = \sqrt{k_1^2 + k_2^2}$ and for the second equation in the lemma, we write

$$\alpha''' = k_1'\mathbf{N}_1 + k_2'\mathbf{N}_2 + k_1^2\mathbf{T} + k_2^2\mathbf{T}$$
$$\alpha' \times \alpha'' = T \times (k_1\mathbf{N}_1 + k_2\mathbf{N}_2) = k_1\mathbf{N}_2 - k_2\mathbf{N}_1$$

in the formula $\tau = \dfrac{\langle \alpha''', \alpha' \times \alpha'' \rangle_L}{\|\alpha' \times \alpha''\|^2}$. Therefore,

$$\tau = \frac{-k_1'k_2 + k_1k_2'}{\|k_1\mathbf{N}_2 + k_2\mathbf{N}_1\|^2} = \frac{k_1k_2' - k_2k_1'}{k_1^2 + k_2^2} = \left(\arctan\frac{k_2}{k_1}\right)'.$$

■

Also, we can state the following lemma for a spacelike curve as follows:

Lemma 5 *Let α be a unit speed spacelike curve then*

$$\kappa(s) = \sqrt{\epsilon_{\mathbf{N}_1}k_1^2 + \epsilon_{\mathbf{N}_2}k_2^2} \quad \text{and} \quad \tau(s) = \epsilon_{\mathbf{N}_1}\theta'(s), \theta(s) = arctanh\frac{k_2}{k_1}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion functions of the curve and k_1, k_2 are principal curvatures along the $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively.

Corollary 6 *k_1 and k_2 correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)\, ds$.*

Corollary 7 *A unit speed non-lightlike curve in Minkowski 3-space lies in a plane if and only if its normal development lies on a line through the origin.*

Proof. It is well known that a curve lies in a plane if and only if torsion function of the curve is zero ($\tau(s) = 0$). So, it means that $\theta' = 0$. On the other hand, $\kappa(s) = \sqrt{\epsilon_{\mathbf{N}_1}k_1^2 + \epsilon_{\mathbf{N}_2}k_2^2} = |k_1\mathbf{N}_1 + k_2\mathbf{N}_2|$. Thus, normal development (k_1, k_2) has θ =constant. That is, lies on a line through the origin. ■

In additionally, parallel frame of a timelike curve in Minkowski $n-$space can be expressed as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \\ \vdots \\ \mathbf{N}_{n-1}' \\ \mathbf{N}_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 & ... & k_n \\ k_1 & 0 & 0 & 0 & ... & 0 \\ k_2 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ k_{r-1} & 0 & 0 & 0 & ... & 0 \\ k_n & 0 & 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \\ \mathbf{N}_n \end{bmatrix}$$

and parallel frame of a spacelike curve in Minkowski n-space can be expressed as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \\ \vdots \\ \mathbf{N}_{n-1}' \\ \mathbf{N}_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 & ... & k_n \\ -(\epsilon_{N_1})k_1 & 0 & 0 & 0 & ... & 0 \\ -(\epsilon_{N_2})k_2 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ -(\epsilon_{N_{n-1}})k_{r-1} & 0 & 0 & 0 & ... & 0 \\ -(\epsilon_{N_n})k_n & 0 & 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \\ \mathbf{N}_n \end{bmatrix}.$$

4 Osculating sphere of a non-lightlike curve in Minkowski 3-space

Now, we will determine osculating sphere of a non-lightlike curve using its parallel frame. Let α be a non-lightlike curve in $\mathbb{E}_1^3$. If we take a function $F(t)$ as

$$F(t) = \langle \alpha(t) - c, \alpha(t) - c \rangle_L \pm r^2$$

where r and c are radius and center of the osculating sphere respectively. Since, the osculating sphere must be 4-point contact with α, then we have

$$F(t) = F'(t) = F''(t) = F'''(t) = 0$$

If we use $\mathbf{T}' = k_1\mathbf{N}_1 + k_2\mathbf{N}_2$ for a non-lightlike curve and $(\langle \mathbf{N}_1, \alpha(t) - c \rangle_L)' = (\langle \mathbf{N}_2, \alpha(t) - c \rangle_L)' = 0$ we find

$$\left.\begin{array}{l} F'(t) = 0 \Longrightarrow \mathbf{T} \perp \alpha(t) - c \\ F''(t) = 0 \Longrightarrow k_1 \langle \mathbf{N}_1, \alpha(t) - c \rangle_L + k_2 \langle \mathbf{N}_2, \alpha(t) - c \rangle_L = -\epsilon_{\mathbf{T}} \\ F'''(t) = 0 \Longrightarrow k_1' \langle \mathbf{N}_1, \alpha(t) - c \rangle_L + k_2' \langle \mathbf{N}_2, \alpha(t) - c \rangle_L = 0 \end{array}\right\} \tag{7}$$

If we say $(\epsilon_{\mathbf{N}_1})\, a = \langle \mathbf{N}_1, \alpha(t) - c \rangle_L$ and $(\epsilon_{\mathbf{N}_2})\, b = \langle \mathbf{N}_2, \alpha(t) - c \rangle_L$ in the equations (7) such that a and b are constant from above, we obtain

$$\begin{array}{l} k_1 (\epsilon_{\mathbf{N}_1})\, a + (\epsilon_{\mathbf{N}_2})\, k_2 b = -\epsilon_T \\ k_1' (\epsilon_{\mathbf{N}_1})\, a + (\epsilon_{\mathbf{N}_2})\, k_2' b = 0 \end{array} \tag{8}$$

From the solution of this linear equation system, we find $a = -\dfrac{(\epsilon_{\mathbf{T}})(\epsilon_{\mathbf{N}_1})\, k_2'}{k_1 k_2' - k_2 k_1'}$ and $b = \dfrac{(\epsilon_{\mathbf{T}})(\epsilon_{\mathbf{N}_2})\, k_1'}{k_1 k_2' - k_2 k_1'}$. So, from $\alpha(s) - c = a\mathbf{N}_1 + b\mathbf{N}_2$, the center and radius of the osculating sphere are found as

$$c(s) = \alpha(s) + \frac{(\epsilon_{\mathbf{T}})(\epsilon_{\mathbf{N}_1})\, k_2'}{k_1 k_2' - k_2 k_1'}\mathbf{N}_1 - \frac{(\epsilon_{\mathbf{T}})(\epsilon_{\mathbf{N}_2})\, k_1'}{k_1 k_2' - k_2 k_1'}\mathbf{N}_2 \tag{9}$$

and

$$r = \sqrt{\frac{(\epsilon_{\mathbf{N}_1})\, k_1'^2 + (\epsilon_{\mathbf{N}_2})\, k_2'^2}{(k_1 k_2' - k_2 k_1')^2}} \tag{10}$$

respectively. Thus, we have proved the following theorem.

Theorem 8 *Let α be a unit speed non-lightlike curve, then the radius and the center of the osculating sphere of α at $\alpha(t)$ are*

$$r = \sqrt{\frac{(\epsilon_{\mathbf{N}_1})\, k_1'^2 + (\epsilon_{\mathbf{N}_2})\, k_2'^2}{(k_1 k_2' - k_2 k_1')^2}} \text{ and } c = \alpha(t) - \frac{(\epsilon_{\mathbf{T}})(\epsilon_{\mathbf{N}_1})\, k_2'}{k_1 k_2' - k_2 k_1'}\mathbf{N}_1 - \frac{(\epsilon_{N_2})\, k_1'}{k_1 k_2' - k_2 k_1'}\mathbf{N}_2$$

where and $\mathbf{N}_1$ and $\mathbf{N}_2$ are parallel vector fields along the curve α.

Corollary 9 *Let α be a unit speed non-lightlike curve in the $\mathbb{E}_1^3$. α has an osculating sphere at $\alpha(s)$ if and only if $k_1 k_2' \neq k_2 k_1'$.*

Also, we can state following corollary from (8) related to normal development (k_1, k_2) of the curve.

Corollary 10 *A unit speed non-lightlike curve is spherical if and only if its normal development (k_1, k_2) lies on a line not through the origin. Also, the distance of this line from the origin is radius of the sphere since*
$r^2 = \langle \alpha(s) - c, \alpha(s) - c \rangle_L = (\epsilon_{\mathbf{N}_1})\, a^2 + (\epsilon_{\mathbf{N}_2})\, b^2$.

Lemma 11 *Let α be a unit speed non-lightlike curve in $\mathbb{E}_1^3$ and the center of osculating sphere at $\alpha(t)$ be c then,*

$$\left\langle \alpha'''(t), \alpha(t) - c \right\rangle_L = 0.$$

Proof. From the derivative equations of a non-lightlike curve according to parallel frame, we have $\alpha''' = k_1'\mathbf{N}_1 + k_2'\mathbf{N}_2 + \kappa^2\mathbf{T}$. Therefore,

$$\left\langle \alpha'''(t), \alpha(t) - c \right\rangle_L = -\frac{(\epsilon_{\mathbf{T}})\, k_2' k_1'}{k_1 k_2' - k_2 k_1'} + \frac{(\epsilon_{\mathbf{T}})\, k_2' k_1'}{k_1 k_2' - k_2 k_1'} = 0.$$

■

5 Parallel frames for non-lightlike curves on the non-lightlike surfaces

5.1 Timelike curve on a timelike surface

Theorem 12 *Given a timelike curve $\alpha : I \to M$ parametrized by arc length on a timelike surface M, suppose that k_1 and k_2 are principal curvatures along the spacelike normal vectors fields $\mathbf{N}_1$ and $\mathbf{N}_2$ of the curve α respectively, then, we have,*

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ k_g & 0 & t_g \\ k_n & -t_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{g} \end{bmatrix}$$

where k_g, k_n and t_g are given by $(k_1\cos\varphi + k_2\sin\varphi)$, $(k_1\sin\varphi - k_2\cos\varphi)$ and $\dfrac{d\varphi}{ds}$ respectively.

Proof. Let M be a timelike surface with the unit spacelike normal vector $\mathbf{N}$ in $\mathbb{E}_1^3$ and α be a timelike curve on $M \subset \mathbb{E}_1^3$, then we have parallel frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ and Darboux frame $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ where spacelike vector $\mathbf{g} = \mathbf{T} \times_L \mathbf{n}$ along the curve α.
Let φ be angle between $\mathbf{N}_1$ and $\mathbf{n}$. In this case, there are the relations (11) and (12) between parallel frame and Darboux frame.

$$\left.\begin{aligned} \mathbf{n} &= \mathbf{N}_1\cos\varphi - \mathbf{N}_2\sin\varphi \\ \mathbf{g} &= \mathbf{N}_1\sin\varphi + \mathbf{N}_2\cos\varphi \end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned} \mathbf{N}_1 &= \mathbf{n}\cos\varphi + \mathbf{g}\sin\varphi \\ \mathbf{N}_2 &= -\mathbf{n}\sin\varphi + \mathbf{g}\cos\varphi \end{aligned}\right\}. \tag{12}$$

By differentiating (11) and using (12) and derivative formulas of parallel frame, we obtain (13).

$$\left.\begin{aligned} \frac{d\mathbf{T}}{dt} &= (k_1\cos\varphi - k_2\sin\varphi)\,\mathbf{n} + (k_1\sin\varphi + k_2\cos\varphi)\,\mathbf{g} \\ \frac{d\mathbf{n}}{dt} &= (k_1\cos\varphi - k_2\sin\varphi)\,\mathbf{T} - \frac{d\varphi}{ds}\mathbf{g} \\ \frac{d\mathbf{g}}{dt} &= (k_1\sin\varphi + k_2\cos\varphi)\,\mathbf{T} + \frac{d\varphi}{ds}\mathbf{n} \end{aligned}\right\} \tag{13}$$

If we represent $(k_1\cos\varphi + k_2\sin\varphi)$, $(k_1\sin\varphi - k_2\cos\varphi)$ and $\dfrac{d\varphi}{ds}$ with the symbols k_n, k_g and t_g respectively, we prove the theorem. ■
Also, the Darboux vector for $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ in terms of the k_n, k_g and t_g is $\mathbf{w} = t_g\mathbf{T} - k_n\mathbf{n} + k_g\mathbf{g}$ since $\dfrac{d\mathbf{T}}{dt} = \mathbf{w} \wedge \mathbf{T}$, $\dfrac{d\mathbf{n}}{dt} = \mathbf{w} \wedge \mathbf{n}$ and $\dfrac{d\mathbf{g}}{dt} = \mathbf{w} \wedge \mathbf{g}$

5.2 Spacelike curve on a spacelike surface

Let M be a spacelike surface with the unit timelike normal vector $\mathbf{n}$ in $\mathbb{E}_1^3$. If α is a spacelike curve on $M \subset \mathbb{E}_1^3$, then we have parallel frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ and Darboux frame $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ with spacelike vector $\mathbf{g} = \mathbf{T} \times_L \mathbf{n}$ along the curve α. Since α is a spacelike curve, $\mathbf{T}$ is a spacelike vector and one of the vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ must be timelike. So, we will examine relations between these frames of the curve with respect to first normal $\mathbf{N}_1$ of the spacelike curve is timelike or not. Let φ be the angle between $\mathbf{N}_1$ and $\mathbf{n}$ respectively.

In the case N_1 is timelike, the relationships between $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ and $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ are as follows:

$$\left.\begin{array}{l} \mathbf{N}_1 = \mathbf{n}\cosh\theta + \mathbf{g}\sinh\theta \\ \mathbf{N}_2 = \mathbf{n}\sinh\theta + \mathbf{g}\cosh\theta \end{array}\right\} \tag{14}$$

$$\left.\begin{array}{l} \mathbf{n} = \mathbf{N}_1\cosh\theta - \mathbf{N}_2\sinh\theta \\ \mathbf{g} = -\mathbf{N}_1\sinh\theta + \mathbf{N}_2\cosh\theta \end{array}\right\}. \tag{15}$$

In the case $\mathbf{N}_2$ is timelike, relations between frames become

$$\left.\begin{array}{l} \mathbf{N}_1 = \mathbf{n}\sinh\theta + \mathbf{g}\cosh\theta \\ \mathbf{N}_2 = \mathbf{n}\cosh\theta + \mathbf{g}\sinh\theta \end{array}\right\} \tag{16}$$

and

$$\left.\begin{array}{l} \mathbf{n} = -\mathbf{N}_1\sinh\theta + \mathbf{N}_2\cosh\theta \\ \mathbf{g} = \mathbf{N}_1\cosh\theta - \mathbf{N}_2\sinh\theta \end{array}\right\}. \tag{17}$$

So, Differentiating $\mathbf{g}$ and $\mathbf{n}$ in (17), (15) and using (14), (17) we state the following theorem

Theorem 13 *Given a spacelike curve $\alpha : I \to M$ parametrized by arc length on a spacelike surface M, suppose that k_1 and k_2 are principal curvatures along the vector fields $\mathbf{N}_1$ and $\mathbf{N}_2$ of the curve α respectively, then,*
i) If $\mathbf{N}_1$ is timelike, then,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ k_n & 0 & -t_g \\ -k_g & -t_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{bmatrix}$$

and Darboux vector is $\vec{w} = -t_g\mathbf{T} - k_g\mathbf{n} - k_n\mathbf{g}$
ii) If $\mathbf{N}_2$ is timelike, then,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ k_g & 0 & -t_g \\ -k_n & -t_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{bmatrix}$$

and Darboux vector is $\vec{w} = -t_g\mathbf{T} - k_n\mathbf{n} - k_g\mathbf{g}$ where k_g, k_n and t_g are given by

$$(k_1\cosh\varphi + k_2\sinh\varphi), \quad (k_1\sinh\varphi + k_2\cosh\varphi) \quad \text{and} \quad \frac{d\varphi}{ds}$$

respectively.

5.3 Spacelike curve on a timelike surface

Let M be a timelike surface with the unit spacelike normal vector $\mathbf{n}$ in $\mathbb{E}_1^3$. If α is a spacelike curve on $M \subset \mathbb{E}_1^3$, then we have the parallel frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ and Darboux frame $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ with timelike vector $\mathbf{g} = -\mathbf{T} \times_L \mathbf{n}$ along the curve α. Since α is a spacelike curve, $\mathbf{T}$ is a spacelike vector and one of the vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ must be timelike. So, similar to above,

we will examine relations between these frames of the curve with respect to first normal $\mathbf{N}_1$of the spacelike curve is timelike or not. Let φ be the angle between N and n respectively.

In the case $\mathbf{N}_1$is timelike, the relationships between $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ and $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ are as follows:

$$\left.\begin{array}{l} \mathbf{N}_1 = \mathbf{g}\cosh\theta + \mathbf{n}\sinh\theta \\ \mathbf{N}_2 = \mathbf{g}\sinh\theta + \mathbf{n}\cosh\theta \end{array}\right\} \tag{18}$$

and

$$\left.\begin{array}{l} \mathbf{g} = \mathbf{N}_1\cosh\theta - \mathbf{N}_2\sinh\theta \\ \mathbf{n} = -\mathbf{N}_1\sinh\theta + \mathbf{N}_2\cosh\theta \end{array}\right\} \tag{19}$$

In the case N_2 is timelike and the relations are

$$\left.\begin{array}{l} \mathbf{N}_2 = \mathbf{g}\cosh\varphi + \mathbf{n}\sinh\varphi \\ \mathbf{N}_1 = \mathbf{g}\sinh\varphi + \mathbf{n}\cosh\varphi \end{array}\right\} \tag{20}$$

and

$$\left.\begin{array}{l} \mathbf{g} = \mathbf{N}_2\cosh\varphi - \mathbf{N}_1\sinh\varphi \\ \mathbf{n} = -\mathbf{N}_2\sinh\varphi + \mathbf{N}_1\cosh\varphi \end{array}\right\} \tag{21}$$

So,we can express the following theorem for these cases

Theorem 14 *Given a spacelike curve $\alpha : I \to M$ parametrized by arc length on a timelike surface M, suppose that k_1 and k_2 are principal curvatures along the vector fields $\mathbf{N}_1$ and $\mathbf{N}_2$ of the curve α respectively, then,*
i) If $\mathbf{N}_1$ is timelike, then,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & -t_g \\ k_n & -t_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{bmatrix}$$

and Darboux vector is $\vec{w} = t_g\mathbf{T} + k_n\mathbf{n} + k_g\mathbf{g}$
ii) If $\mathbf{N}_2$ is timelike, then,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ -k_n & 0 & -t_g \\ k_g & -t_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{bmatrix}$$

and Darboux vector is $\vec{w} = t_g\mathbf{T} + k_g\mathbf{n} + k_n\mathbf{g}$ where k_g, k_n and t_g are given by

$$\left(k_1\cosh\varphi + k_2\sinh\varphi\right), \quad \left(k_1\sinh\varphi + k_2\cosh\varphi\right) \quad \text{and} \quad \frac{d\varphi}{ds}$$

respectively.

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