Advanced Mathematical Models & Applications
Vol.11, No.2, 2026, pp.414-427
https://doi.org/10.62476/amma112414

Jomard Publishing

INEQUALITIES PERTAINING TO GENERAL FRACTIONAL INTEGRAL OPERATORS FOR S-CONVEX FUNCTIONS IN THE THIRD SENSE

Gültekin Tınaztepe[1], **Ilknur Yesilce Isik**[2], **Sinem Sezer Evcan**[1], **Gabil Adilov**[3]

[1]Akdeniz University, Antalya, Turkiye
[2]Department of Mathematics, Faculty of Science and Letters, Aksaray University, Aksaray, Turkiye
[3]Faculty of Engineering, Karabakh University, Khankendi, Azerbaijan

Abstract. This article investigates a novel class of inequalities for s-convex functions in the third sense, utilizing generalized fractional integral operators with respect to another function. We establish two foundational theorems that characterize both left-sided and right-sided fractional integral operators, providing a comprehensive framework for this specific function class. Building upon these core results, we derive a series of theorems that elucidate the intricate relationships between the proposed inequalities. To demonstrate the versatility and scope of our findings, we specialize the results to Riemann-Liouville and Hadamard fractional integral settings. Notably, the theoretical framework is applied to yield significant new inequality relations among key special functions, including Beta and Incomplete Beta functions, Gamma and Incomplete Gamma functions, and Imaginary Error functions, thereby bridging abstract fractional calculus with practical analytic applications. Furthermore, it is demonstrated that one of these inequalities provide significantly sharper bounds than analogue one currently established in the literature.

Keywords: s-convexity in the Third Sense, Hermite-Hadamard Type Inequality, Generalized Fractional Integral.
AMS Subject Classification: 26B25, 52A40, 91B52.
Corresponding author: Gabil Adilov, Faculty of Engineering, Karabakh University, Khankendi, Azerbaijan, *e-mail: qabil.adilov@karabakh.edu.az*
Received: 20 November 2025; Revised: 18 January 2026; Accepted: 10 February 2026; Published: 3 August 2026.

1 Introduction

In recent decades, the tools of fractional calculus have found extensive application across a wide spectrum of scientific disciplines, from engineering to medicine (Sun et al., 2018). This has led to the definition of numerous fractional operators based on various kernel functions, including the Riemann-Liouville, Hadamard, Caputo, Katugampola, conformable, and general types (Kilbas et al., 1993; Katugampola, 2014). Given that convex functions are fundamentally characterized by integral inequalities such as Hermite-Hadamard, Fejér, and Ostrowski, the prevalent use of fractional integrals necessitates the re-examination and reformulation of convexity within the framework of these fractional operators. Consequently, there has been a significant surge in research focused on generalizing classical integral inequalities, particularly the Hermite-Hadamard inequality for convex functions via fractional integrals (Mohammed, 2021; Set et al., 2014; Fernandez & Mohammed, 2021; Jleli et al., 2016; Bohner et al., 2022; Tınaztepe et al., 2021; Dahmani, 2010; Chen & Luo, 2023).

On the other hand, several new generalizations and extensions of convex functions are introduced throughout that time (Adilov & Yesilce, 2017; Briec & Horvath, 2004; Briec et al., 2005; Sezer et al., 2021). Among them are s-convex functions in third and fourth senses recently Kemali et al. (2021); Eken et al. (2021), whose first and second senses are introduced in Breckner (1978); Orlicz (1961). While the second and fourth sense s-convex functions are defined on convex sets, the others are defined on special sets, namely, s-convex sets. Various scholars have examined the fundamental characteristics of these functions via inequalities (Kemali, 2021; Eken, 2021; Sezer, 2021b,a; Eken, 2021). s-convex functions in the third sense are considered in this work. These functions are defined in Kemali et al. (2021). Generally, s-convex functions have some applications in fractal theory (Kılıçman & Saleh, 2015; Kwun et al., 2019).

This study integrates s-convexity in the third sense with generalized fractional integral operators to provide a more refined analytical framework for modeling complex phenomena that transcend the limitations of classical convexity. By leveraging the inherent non-local properties and memory effects of fractional calculus, the research establishes a robust mathematical foundation for extending the theoretical landscape of integral inequalities. Consequently, the derived inequalities offer sharper bounds and enhanced predictive accuracy for essential special functions, facilitating more precise applications in advanced fields such as quantum physics and statistical mechanics.

Many scholars have studied the Hermite-Hadamard inequality in characterization of various types of convex functions (Dragomir & Pearce, 2000; Tınaztepe et al., 2021; Sezer, 2021a,b). Also more generalized versions of some of these inequalities via fractions are presented in Jleli et al. (2016); Tınaztepe et al. (2021). The generalized fractional integral inequalities for s-convex functions in the third sense has not been studied. In this study, we set the Hermite-Hadamard type inequalities via general fractional operator with regard to another function of s-convex functions in the third sense. Also, results are expressed for the Riemann-Liouville and Hadamard fractionals, which are special cases of general fractional integral operator. Finally, some remarkable relations between gamma and incomplete gamma; beta and incomplete are given as application examples of the results. In addition, an upper bound functions in terms of elementary functions for imaginary error function is presented, it is shown that this bound is more precise than the existing one in literature.

2 Preliminaries

The following notations, which will be used throughout the text, should be presented before recalling definitions.

$\mathbb{R}^n$ denotes the n-dimensional real vector space and

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i \in \{1, 2, .., n\}\}.$$

Definition 1. *Kemali et al. (2021) Let $0 < s \leq 1$ and $A \subset \mathbb{R}^n$. The set A is said to be an s-convex set if $\lambda x + \mu y \in A$, for all $x, y \in A$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^s + \mu^s = 1$.*

This definition coincides with the definition of "p-convex set" given in Sezer et al. (2021). As it is said for p-convex sets in Sezer et al. (2021), every interval of real numbers containing zero or accepting it as a boundary point is an s-convex set.

Definition 2. *(Kemali et al., 2021) Let $A \subset \mathbb{R}^n$ be an s-convex set and $f : A \to \mathbb{R}$. The function f is called s-convex function in the third sense if*

$$f(\lambda x + \mu y) \leq \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \tag{1}$$

holds for all $x, y \in A$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^s + \mu^s = 1$.

The inequality (1) can be represented in terms of one parameter $t \in [0,1]$ instead of λ, μ in the following way :

$$f(t^{\frac{1}{s}}x + (1-t)^{\frac{1}{s}}y) \leq t^{\frac{1}{s^2}} f(x) + (1-t)^{\frac{1}{s^2}} f(y).$$

In studies on third sense s-convex functions, $f(x) = -x^{\frac{1}{s}}$ on $\mathbb{R}_+$ is mainly used since it is one of the very few examples known given in Kemali et al. (2021). We suggest a characterization of s-convex functions which enables us to obtain various function examples. To achieve this goal, an auxiliary idea is presented below.

Lemma 1. *Let $[a,b] \subset \mathbb{R}$ with $0 \in [a,b]$. Suppose $f : [a,b] \to \mathbb{R}$ is a convex function with $f(0) \leq 0$. If $\lambda + \mu \leq 1$ for $\lambda, \mu \geq 0$, then for all $x, y \in [a,b]$,*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y).$$

Proof. Let $\rho = \lambda + \mu$. If $\rho = 0$, then $\lambda = \mu = 0$ thus resulting clear inequality. Suppose $\rho > 0$. Hence

$$\lambda x + \mu y = \rho \left(\frac{\lambda}{\rho} x + \frac{\mu}{\rho} y \right) + (1-\rho) \cdot 0.$$

From convexity of f and $f(0) \leq 0$,

$$\begin{aligned} f(\lambda x + \mu y) &\leq \rho f\left(\frac{\lambda}{\rho} x + \frac{\mu}{\rho} y \right) + (1-\rho) f(0) \\ &\leq \rho f\left(\frac{\lambda}{\rho} x + \frac{\mu}{\rho} y \right). \end{aligned} \tag{2}$$

By making use of the convexity again, one has

$$f\left(\frac{\lambda}{\rho} x + \frac{\mu}{\rho} y \right) \leq \frac{\lambda}{\rho} f(x) + \frac{\mu}{\rho} f(y). \tag{3}$$

Multiplication (3) with ρ and consideration of (2) leads to the desired inequality. □

Theorem 1. *Let $[a,b] \subset \mathbb{R}$ with $0 \in [a,b]$. Suppose $f : [a,b] \to \mathbb{R}_-$ is a convex function. Then f is s-convex function in the third sense.*

Proof. Since $t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}} \leq 1$, taking $\lambda = t^{\frac{1}{s}}$ and $\mu = (1-t)^{\frac{1}{s}}$ in Lemma and using $t^{\frac{1}{s^2}} \leq t^{\frac{1}{s}}$ and $f \leq 0$ yields

$$f(t^{\frac{1}{s}}x + (1-t)^{\frac{1}{s}}y) \leq t^{\frac{1}{s}} f(x) + (1-t)^{\frac{1}{s}} f(y) \leq t^{\frac{1}{s^2}} f(x) + (1-t)^{\frac{1}{s^2}} f(y).$$

□

Note that although the condition that f is negative valued function is restrictive, we can find several examples of the s-convex functions in the third sense by means of Theorem 1.

According to Theorem 1, the functions: $-\arctan x + a$ with $a \leq 0$ on $[0, \infty)$; the quadratic functions $x(x-a) + b$ with $b < 0$ on $[0, a]$; any affine function $cx + d$ with $c, d \leq 0$ on $[0, \infty)$; $-a \cos x + c$ with $a \geq 0$ and $c \leq 0$ on $[0, \frac{\pi}{2}]$ are s-convex functions in the third sense.

Note that from now on, we shortly use "s-convex function" to say "s-convex function in the third sense" throughout the rest of paper. Let us reveal some relations which will be used in next sections.

Lemma 2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a decreasing s-convex function. If $a, b \in \mathbb{R}_+$ with $a < b$ and $t \in [0,1]$, then*

$$f\left(\frac{a+b}{2^{\frac{1}{s}}} \left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}} \right) \right) \leq f(\frac{a+b}{2^{\frac{2}{s}-1}}). \tag{4}$$

Proof. For $t \in [0,1]$, a quick derivative calculation shows that $t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}$ arrives the minimum value on $t = \frac{1}{2}$, whereby $\frac{1}{2^{\frac{1}{s}-1}} \leq t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}$. Multiplying both side $\frac{a+b}{2^{\frac{1}{s}}}$, then taking images under f both sides and using monotonicity of f, we have (4). □

Remark 1. *Lemma above can be restated for the functions f defined on $\mathbb{R}$, thus extending the domain of the function from nonnegative real numbers to whole real numbers, by replacing the condition of being decreasing with the following*

$$f(\lambda x) \leq f(x)$$

such that $1 \leq \lambda \leq 2^{\frac{1}{s}-1}$. In this case, taking intıo account that $1 < 2^{\frac{1}{s}-1}(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}) < 2^{\frac{1}{s}-1}$, we can write

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right) = f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\left(2^{\frac{1}{s}-1}(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}})\right)\right) \leq f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)$$

Lemma 3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an s-convex function. Suppose $a, b \in \mathbb{R}$ with $a < b$ and $t \in [0,1]$. If $f \leq 0$, then*

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right) \leq \frac{1}{2^{\frac{2}{s^2}-1}}\left(f(a) + f(b)\right). \tag{5}$$

Proof. Suppose that $f \leq 0$. Using s-convexity, we have

$$\begin{aligned} f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right) &\leq \frac{1}{2^{\frac{1}{s^2}}}\left[f\left(a\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right) + f\left(b\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)\right] \\ &\leq \frac{1}{2^{\frac{1}{s^2}}}\left(t^{\frac{1}{s^2}} + (1-t)^{\frac{1}{s^2}}\right)\left(f(a) + f(b)\right). \end{aligned}$$

In a similar way as in the proof of Lemma 2, it is seen that $\frac{1}{2^{\frac{1}{s^2}-1}} \leq t^{\frac{1}{s^2}} + (1-t)^{\frac{1}{s^2}}$ and minimum value is attained for $t = \frac{1}{2}$. By negativity of f,

$$\left(t^{\frac{1}{s^2}} + (1-t)^{\frac{1}{s^2}}\right)\left(f(a) + f(b)\right) \leq \frac{1}{2^{\frac{1}{s^2}-1}}\left(f(a) + f(b)\right).$$

Then multiplying both sides $\frac{1}{2^{\frac{1}{s^2}}}$ gives the result. □

The general fractional operator with regard to another function is defined as follows:

Definition 3. *Kilbas et al. (1993) Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ such that $f \in L_1[a,b]$. Suppose $g : [a,b] \to \mathbb{R}$ be an increasing monotone function on $(a,b]$ with a continuous derivative $g'(x)$ on (a,b). The left-sided and right-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ is defined by*

$$I^{\alpha}_{a^+;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{[g(x) - g(t)]^{1-\alpha}} dt, \qquad x > a$$

and

$$I^{\alpha}_{b^-;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{[g(t) - g(x)]^{1-\alpha}} dt, \qquad x < b$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx \quad \text{for} \quad \alpha > 0$$

respectively, provided that the integrals exist.

3 Main Results

The generalized fractional integrals will be used to demonstrate the new varieties of Hermite-Hadamard type inequalities. As a result of the fact that these generic fractional integrals can be transformed into the most well-known fractional integrals, novel inequalities involving the Riemann-Liouville and Hadamard integrals can be obtained.

Theorem 2. *Let $\alpha > 0$ and $a, b \in \mathbb{R}_+$ with $a < b$ and $f : \mathbb{R}_+ \to \mathbb{R}_-$ be decreasing s-convex function such that $f \in L_1[0, +\infty)$. Assume that $g : [0, b] \to \mathbb{R}$ is an increasing function on $(0, b]$ with a continuous derivative $g'(x)$ on $(0, b)$. The following left sided fractional inequality holds:*

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+};g} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{1}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha} \min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\}.$$

Proof. Consider $t \in \left[\frac{1}{2}, 1\right]$ which makes the expression $\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1} - (1-t)^{\frac{1}{s}-1}\right)$ positive. Let us multiply both side of (4) with the factor, which is positive,

$$\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1} - (1-t)^{\frac{1}{s}-1}\right)\right), \tag{6}$$

then integrate over t. So

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right) \frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)\right]^{1-\alpha}} \times \\
&\qquad \left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1} - (1-t)^{\frac{1}{s}-1}\right)\right) dt \\
&\leq \int_{\frac{1}{2}}^{1} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)\right)\right]^{1-\alpha}} \times \\
&\qquad \left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1} - (1-t)^{\frac{1}{s}-1}\right)\right) dt\ .
\end{aligned}$$

Making the substitution $u = \frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right)$ and dividing both side of the inequality by $\Gamma(\alpha)$, we have

$$\begin{aligned}
&\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}} \frac{f(u)\,g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g(u)\right]^{1-\alpha}} du \leq f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}} \frac{g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g(u)\right]^{1-\alpha}} du \\
&\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}} \frac{f(u)\,g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g(u)\right]^{1-\alpha}} du \leq f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \frac{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}}{\alpha} \\
&I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+};g} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{7}$$

On the other hand, if (6) is multiplied with both sides of (5) , then the following inequality is obtained:

$$\int_{\frac{1}{2}}^{1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)\right]^{1-\alpha}}\times \left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1}-(1-t)^{\frac{1}{s}-1}\right)\right)dt$$

$$\leq \int_{\frac{1}{2}}^{1}\frac{1}{2^{\frac{2}{s^2}-1}}\left(f(a)+f(b)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)\right]^{1-\alpha}}\times \left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1}-(1-t)^{\frac{1}{s}-1}\right)\right)dt.$$

Making the substitution $u=\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)$ and then multiplying both side with $\frac{1}{\Gamma(\alpha)}$, we have

$$\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g(u)\right]^{1-\alpha}}du\leq\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g(u)\right]^{1-\alpha}}du$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g(u)\right]^{1-\alpha}}du\leq\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}\alpha\Gamma(\alpha)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}$$

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+};g}f\left(\frac{a+b}{2^{\frac{1}{s}}}\right)\leq\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}. \quad (8)$$

Merging (7) and (8) obtained inequalities, one has the result. □

Considering Remark 1 and then obtaining (7), we can extend the theorem above to the functions f without the restrictions of being defined on non-negative real numbers and having negative values.

Corollary 1. *Let $\alpha>0$ and $a,b\in\mathbb{R}$ with $a<b$ and $f:\mathbb{R}\to\mathbb{R}$ be an s-convex function such that $f\in L_1(\mathbb{R})$ and $f(\lambda x)\leq f(x)$ for $1\leq\lambda\leq 2^{\frac{1}{s}-1}$. Assume that $g:\mathbb{R}\to\mathbb{R}$ is an increasing function on $\mathbb{R}$ with a continuous derivative $g'(x)$ on $\mathbb{R}$. The following left sided fractional inequality holds:*

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+};g}f\left(\frac{a+b}{2^{\frac{1}{s}}}\right)\leq\frac{1}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right).$$

Validity of Corollary is based on obtaining (7) in the proof of Theorem 2. It is clear that the condition on $f(\lambda x)\leq f(x)$ for $1\leq\lambda\leq 2^{\frac{1}{s}-1}$ preserves (4), thus enabling us to follow the same proof outline for obtaining (7).

Theorem 3. *Let $a,b\in\mathbb{R}_+$ with $a<b$ and $g:[0,b]\to\mathbb{R}$ be an increasing function with a continuous derivative $g'(x)$ on $(0,b)$. Let $f:\mathbb{R}_+\to\mathbb{R}_-$ be decreasing s-convex function such that $f\in L_1[0,+\infty)$ and $\alpha>0$. The following right-sided general fractional inequality holds:*

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-};g}f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\leq\frac{1}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}\min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}},f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\}$$

Proof. Consider $t \in \left[\frac{1}{2}, 1\right]$. If (4) is multiplied by the expression

$$\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}\left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1}-(1-t)^{\frac{1}{s}-1}\right)\right) \tag{9}$$

and integrated over $t \in \left[\frac{1}{2}, 1\right]$,

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-};g} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{1}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \tag{10}$$

is obtained. On the other hand, if we multiply (5) with (9) and integrate both sides over $t \in \left[\frac{1}{2}, 1\right]$, then

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}\times \\
&\qquad\left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1}-(1-t)^{\frac{1}{s}-1}\right)\right)dt \\
&\leq \int_{\frac{1}{2}}^{1}\frac{1}{2^{\frac{2}{s^2}-1}}\left(f(a)+f(b)\right)\frac{g'\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}\times \\
&\qquad\left(\frac{a+b}{2^{\frac{1}{s}}s}\left(t^{\frac{1}{s}-1}-(1-t)^{\frac{1}{s}-1}\right)\right)dt
\end{aligned}$$

is obtained. Making the substitution $u = \frac{a+b}{2^{\frac{1}{s}}}\left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)$, we have

$$\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{f(u)g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}du \leq \frac{1}{2^{\frac{2}{s^2}-1}}\left(f(a)+f(b)\right)\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}du$$

$$\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{f(u)g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}du \leq \frac{1}{2^{\frac{2}{s^2}-1}}\frac{1}{\alpha}\left(f(a)+f(b)\right)\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2^{\frac{2}{s}-1}}}^{\frac{a+b}{2^{\frac{1}{s}}}}\frac{f(u)g'(u)}{\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{1-\alpha}}du \leq \frac{1}{2^{\frac{2}{s^2}-1}}\frac{f(a)+f(b)}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}$$

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-};g} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{1}{2^{\frac{2}{s^2}-1}}\frac{f(a)+f(b)}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha}$$

Eventually, combination of obtained inequalities gives the required result. □

Considering the Remark 1 and then (7), we can extend the theorem above to the functions f without the restrictions of being defined on non-negative real numbers and having negative values.

Corollary 2. *Let $\alpha > 0$ and $a, b \in \mathbb{R}$ with $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ be an s-convex function such that $f \in L_1(\mathbb{R})$. Assume that $g : \mathbb{R} \to \mathbb{R}$ is an increasing function on $\mathbb{R}$ with a continuous derivative $g'(x)$ on $\mathbb{R}$. Suppose $f(\lambda x) \leq f(x)$ for $1 \leq \lambda \leq 2^{\frac{1}{s}-1}$. The following left sided fractional inequality holds:*

$$I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-};g} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{1}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right).$$

Validity of Corollary is based on obtaining (10) in the proof of Theorem 3. It is clear that the condition on $f(\lambda x) \leq f(x)$ for $1 \leq \lambda \leq 2^{\frac{1}{s}-1}$ preserves (4), thus enabling us to follow the same proof outline for obtaining (10).

Combining both theorems above, we can present a Hermite-Hadamard type inequality as follows.

Corollary 3. *Under the conditions of Theorem 2, the following inequality holds:*

$$\begin{aligned} &I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+};g} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) + I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-};g} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \\ &\leq \frac{2}{\Gamma(\alpha+1)}\left[g\left(\frac{a+b}{2^{\frac{1}{s}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right]^{\alpha} \min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\} \end{aligned}$$

In case $g(x) = x$ in Definition 3, the Riemann-Liouville fractional integrals are obtained. The left-sided and right-sided Riemann-Liouville integrals of order $\alpha > 0$ is denoted as follows

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b$$

respectively, provided that the integrals exist.

By Theorem 2 and 3, the following inequality for the Riemann-Liouville integrals of s-convex functions are obtained.

Theorem 4. *Let $a, b \in \mathbb{R}_+$ with $a < b$ and $\alpha > 0$. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_-$ is decreasing s-convex function such that $f \in L_1[0, +\infty)$. Then*

$$\begin{aligned} &\max\left\{J^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right), J^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right)\right\} \\ &\leq \frac{(a+b)^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{2^{\frac{1}{s}}-2}{2^{\frac{2}{s}}}\right]^{\alpha} \min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\} \end{aligned}$$

Proof. Accepting $g(x) = x$ in Theorem 2 and Theorem 3 yields

$$J^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{(a+b)^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{2^{\frac{1}{s}}-2}{2^{\frac{2}{s}}}\right]^{\alpha} \min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\} \tag{11}$$

and

$$J^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{(a+b)^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{2^{\frac{1}{s}}-2}{2^{\frac{2}{s}}}\right]^{\alpha} \min\left\{\frac{f(a)+f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\right\}. \tag{12}$$

respectively. □

Hadamard fractional integrals are another one of the specific essential fractional integrals produced by the general fractional integral operator described in this article. In case $g(x) = \ln(x)$ in Definition 3, the Hadamard fractional integrals are obtained. The left-sided and right-sided Hadamard fractional integral of order $\alpha > 0$ of f is defined by

$$\mathbf{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x > a$$

and

$$\mathbf{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x < b,$$

respectively, provided that the integrals exist.

The following theorems provide the inequality for s-convex functions with Hadamard fractional integral operators.

Theorem 5. *Let $a, b \in \mathbb{R}_+$ with $a < b$ and $\alpha > 0$. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_-$ is decreasing s-convex function such that $f \in L_1[0, +\infty)$ and $\alpha > 0$. The following inequality involving Hadamard fractional integral holds*

$$\max\left\{ \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^+} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right), \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^-} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \right\}$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\left(\frac{1}{s} - 1\right) \ln 2\right]^{\alpha} \min\left\{ \frac{f(a) + f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \right\}$$

Proof. Considering $g(x) = \ln x$ in Theorem 2 and Theorem 3, which is increasing function on $(a, b]$ with a continuous derivative $g'(x) = \frac{1}{x}$ on (a, b) yields

$$\mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^+} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{1}{\Gamma(\alpha+1)} \left[\left(\frac{1}{s} - 1\right) \ln 2\right]^{\alpha} \min\left\{ \frac{f(a) + f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \right\} \tag{13}$$

and

$$\mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^-} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{1}{\Gamma(\alpha+1)} \left[\left(\frac{1}{s} - 1\right) \ln 2\right]^{\alpha} \min\left\{ \frac{f(a) + f(b)}{2^{\frac{2}{s^2}-1}}, f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \right\} \tag{14}$$

respectively. Combining these inequalities gives the result. □

4 Applications

Using the main results, we can obtain some relations between gamma and incomplete gamma functions; beta and incomplete beta functions. Also an upper bound for imaginary error function is obtained.

Theorem 6. *Let $x, \alpha \geq 0$. Then*

$$\beta(x+2, \alpha) - \beta_{2^{-x}}(x+2, \alpha) \geq \frac{1}{\alpha} \left(1 - 2^{-x}\right)^{\alpha} \frac{1}{2^{x^2+x}}$$

where β and β_* denotes beta and incomplete beta respectively, i.e.

$$\beta(w_1, w_2) = \int_0^1 x^{w_1-1} (1-x)^{w_2-1} dx \quad \text{for } 0 < w_1, w_2,$$
$$\beta_k(w_1, w_2) = \int_0^k x^{w_1-1} (1-x)^{w_2-1} dx \quad \text{for } 0 < w_1, w_2 \text{ and } 0 < k \leq 1$$

Proof. Consider $f(x) = -x^{\frac{1}{s}}$ in (11). By Mathematica 13.1, we have

$$J^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}}-1}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) = \frac{1}{\Gamma(\alpha)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{\alpha+\frac{1}{s}}\left(\beta_{2^{1-\frac{1}{s}}}(\frac{1}{s}+1,\alpha) - \beta(\frac{1}{s}+1,\alpha)\right)$$

thus,

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{\alpha+\frac{1}{s}}\left(\beta_{2^{1-\frac{1}{s}}}(\frac{1}{s}+1,\alpha) - \beta(\frac{1}{s}+1,\alpha)\right)\\ &\leq \frac{(a+b)^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{2^{\frac{1}{s}}-2}{2^{\frac{2}{s}}}\right]^{\alpha}\min\left\{-\frac{a^{\frac{1}{s}}+b^{\frac{1}{s}}}{2^{\frac{2}{s^2}-1}}, -\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}\right\}.\end{aligned}$$

After some simplifications,

$$\left(\frac{a+b}{2^{\alpha+\frac{1}{s}}}\right)^{\frac{1}{s}}\left(\beta(\frac{1}{s}+1,\alpha) - \beta_{2^{1-\frac{1}{s}}}(\frac{1}{s}+1,\alpha)\right) \geq \frac{1}{\alpha}\left[\frac{2^{\frac{1}{s}}-2}{2^{\frac{2}{s}}}\right]^{\alpha}\max\left\{\frac{a^{\frac{1}{s}}+b^{\frac{1}{s}}}{2^{\frac{2}{s^2}-1}}, \left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}\right\}.$$

Since $a^{\frac{1}{s}}+b^{\frac{1}{s}} \leq (a+b)^{\frac{1}{s}}$ and $\frac{1}{2^{\frac{2}{s^2}-1}} \leq \frac{1}{2^{\frac{2}{s^2}-\frac{1}{s}}}$ for $s \in (0,1]$, it is clear that $\frac{a^{\frac{1}{s}}+b^{\frac{1}{s}}}{2^{\frac{2}{s^2}-1}} \leq \left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}$. Therefore

$$\max\left\{\frac{a^{\frac{1}{s}}+b^{\frac{1}{s}}}{2^{\frac{2}{s^2}-1}}, \left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}\right\} = \left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}.$$

Simplifying $(a+b)^{\frac{1}{s}}$ in the inequality, we have

$$\beta(\frac{1}{s}+1,\alpha) - \beta_{2^{1-\frac{1}{s}}}(\frac{1}{s}+1,\alpha) \geq \frac{1}{\alpha}\left(\frac{2^{\frac{1}{s}-1}-1}{2^{\frac{1}{s}-1}}\right)^{\alpha}\frac{1}{2^{\frac{1}{s^2}-\frac{1}{s}}}.$$

Taking $x = \frac{1}{s} - 1$, whereby $x \geq 0$, we get the result. □

Theorem 7. *Let $x, \alpha \geq 0$. Then*

$$\beta(-x-1-\alpha,\alpha) - \beta_{2^{-x}}(-x-1-\alpha,\alpha) \geq \frac{(2^x-1)^{\alpha}}{\alpha}.$$

Proof. Application of same ideas in the proof above to (12) yields to result. Note that although beta and incomplete beta function is defined on positive real numbers, it can be defined for negative real numbers by mean of analytic continuation. □

Theorem 8. *Let $\alpha \geq 0$ and $x \geq 1$. Then*

$$\Gamma(\alpha) - \Gamma(\alpha, \ln x) \geq \frac{1}{\alpha}\frac{\ln^{\alpha} x}{x}.$$

where Γ and $\Gamma(\cdot,\cdot)$ denotes gamma and incomplete gamma respectively, i.e.

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x}x^{\alpha-1}dx \quad \textit{for} \;\; \alpha > 0,$$
$$\Gamma(\alpha, z) = \int_z^{\infty} e^{-x}x^{\alpha-1}dx \quad \textit{for} \;\; \alpha, z > 0,$$

Proof. Consider $f(x) = -x^{\frac{1}{s}}$ in (13). By Mathematica 13.1, we have

$$\mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{+}} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) = -\frac{s^{\alpha}}{\Gamma(\alpha)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}}\left(\Gamma(\alpha) - \Gamma(\alpha, \left(\frac{1}{s^2} - \frac{1}{s}\right)\ln 2)\right)$$

thus,

$$\begin{aligned} &-\frac{s^{\alpha}}{\Gamma(\alpha)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}}\left(\Gamma(\alpha) - \Gamma(\alpha, \left(\frac{1}{s^2} - \frac{1}{s}\right)\ln 2)\right) \\ &\leq \frac{1}{\Gamma(\alpha+1)}\left[\left(\frac{1}{s} - 1\right)\ln 2\right]^{\alpha} \min\left\{-\frac{a^{\frac{1}{s}} + b^{\frac{1}{s}}}{2^{\frac{2}{s^2}-1}}, -\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}\right\}. \end{aligned}$$

Using the similar simplifications and facts in the proof of Theorem 6, we have

$$\Gamma(\alpha) - \Gamma(\alpha, \left(\frac{1}{s^2} - \frac{1}{s}\right)\ln 2) \geq \frac{1}{\alpha}\left[\frac{1}{s}\left(\frac{1}{s} - 1\right)\ln 2\right]^{\alpha}\left(\frac{1}{2^{\frac{1}{s}-1}}\right)^{\frac{1}{s}}.$$

Accepting $2^{\frac{1}{s^2}-\frac{1}{s}} = x$ whence $x \geq 1$, we get the result. □

Theorem 9. *Let $x \geq 0$. Then*

$$\operatorname{erfi}(x) \leq \frac{1}{\sqrt{\pi}}\left(2xe^{x^2} - \frac{4x^3}{3}\right)$$

where erfi *denotes imaginary error function, i.e.*

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt.$$

Proof. Consideration of $f(x) = -x^{\frac{1}{s}}$ and $\alpha = \frac{3}{2}$ in (14) yields

$$\mathbf{J}^{\frac{3}{2}}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq -\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left[\left(\frac{1}{s} - 1\right)\ln 2\right]^{\frac{3}{2}}\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)^{\frac{1}{s}}.$$

By Mathematica 13.1, we have

$$\mathbf{J}^{\frac{3}{2}}_{\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{-}} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) = -\frac{1}{\frac{1}{2}\sqrt{\pi}}\frac{s}{2}\left(\frac{a+b}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}}\left(2\sqrt{\ln 2^{\frac{1}{s}-1}} - \sqrt{\pi s}\left(2^{\frac{1}{s}-1}\right)^{-\frac{1}{s}}\operatorname{erfi}(\sqrt{\ln 2^{\frac{1}{s^2}-\frac{1}{s}}})\right).$$

Taking into account that $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma(\frac{3}{2})$ and making easy simplifications give

$$2^{\frac{1}{s^2}-\frac{1}{s}+1}\sqrt{\ln 2^{\frac{1}{s}-1}} - \sqrt{\pi s}\operatorname{erfi}(\sqrt{\ln 2^{\frac{1}{s^2}-\frac{1}{s}}}) \geq \frac{4}{3s}\left[\ln 2^{\frac{1}{s}-1}\right]^{\frac{3}{2}}.$$

The substitution $x = \sqrt{\ln 2^{\frac{1}{s^2}-\frac{1}{s}}}$ where $x \geq 0$ yields the required result. □

As seen above, the results obtained in applications are based on only one example of s-convex functions in third sense. Usage of Theorem 1, thus obtaining new examples of the s-convex functions, enables the derivation of numerous additional results.

To demonstrate the effectiveness of the results obtained, we compare them with some relevant inequalities in the literature. The concept of an inequality being "sharper" (more precise) than another can be defined as follows:

For $x \in A \subset \mathbb{R}$, let $g(x) > h(x)$ be given. If there exists a non-negative function $u(x)$ such that $g(x) - h(x) > u(x)$ for all $x \in A$, then the inequality $g(x) - h(x) > u(x)$ is said to be sharper than the inequality $g(x) > h(x)$.

Demonstrating this algebraically is often difficult. To perform a comparison, the inequalities must have the same direction and the term in one side must be identical. Based on this principle, the obtained inequalities can be numerically compared with relevant ones found in the literature. However, it may not always be possible to reach a definitive conclusion.

Let us consider the following inequality involving gamma and incomplete gamma function, provided in Corollary 3.4 by Yang et al. (2016):

Let $\alpha \in (0,1), k \geq 2$, and $x > 0$. Then the inequality

$$\frac{(x+k)^\alpha - x^\alpha}{\alpha k} < e^x \Gamma(\alpha, x) < \frac{\Gamma(\alpha)[(x+k)^\alpha - x^\alpha]}{k^\alpha} \tag{15}$$

is valid.

To compare the right part of (15) with the suggested inequality in Theorem 8, we arrange the inequalities so that they have the same direction and the same term on the same side as follows:

$$\Gamma(\alpha, x) < \frac{\Gamma(\alpha)[(x+k)^\alpha - x^\alpha]}{k^\alpha} e^{-x} := w_1(\alpha, k, x)$$
$$\Gamma(\alpha, x) \leq \Gamma(\alpha) - \frac{1}{\alpha} x^\alpha e^{-x} := w_2(\alpha, k, x).$$

If we take $\alpha = 0.6, k = 3$, and $x = 4$, it is observed that $w_1 = 0.01293 < w_2 = 1.4191$. This indicates that the right part of (15) is sharper than suggested inequality for these values. However, if we take $\alpha = 0.6, k = 3$, and $x = 0.01$, then $w_1 = 1.4289 > w_2 = 1.3851$. This shows the contrary, i.e. the suggested inequality is sharper than the right part of (15) for these values. Moreover, numerical experiments indicate that the suggested inequality is sharper for x values very close to zero.

In Tınaztepe et al. (2025), the following bound regarding the imaginary error function ($erfi$) is provided:

Let $0 \leq y < x$. Then

$$erfi(x) - erfi(y) \leq \frac{1}{\sqrt{\pi}} \left[6(y+x) e^{y^2+x^2} + 2\frac{(e^{x^2} - e^{y^2})}{y+x} \right].$$

By setting $y = 0$, we obtain

$$erfi(x) \leq \frac{1}{\sqrt{\pi}} \left[6x e^{x^2} + 2\frac{(e^{x^2} - 1)}{x} \right].$$

In comparison to the suggested inequality in Theorem 9, since

$$2x e^{x^2} - \frac{4x^3}{3} < 6x e^{x^2} + 2\frac{(e^{x^2} - 1)}{x}$$

holds for $x > 0$, this inequality provides a smaller upper bound for $erfi$. Therefore, the proposed inequality is sharper.

The inequality regarding the Beta function can be analyzed in a similar manner.

Acknowledgements

We are grateful to Akdeniz University, Aksaray University and Karabakh University for support of this research.

References

Adilov, G., Yesilce, I. (2017). $\mathbb{B}^{-1}$-convex Functions. *Journal of Convex Analysis*, *24*(2), 505–517.

Briec, W., Horvath, C. (2004). $\mathbb{B}$ -convexity. *Optimization*, *53*, 103-127.

Briec, W., Horvath, C. & Rubinov, A. (2005). Separation in $\mathbb{B}$-convexity. *Pacific Journal of Optimization*, *1*(1), 13-30.

Bohner, M., Kashuri, A., Mohammed, P. & Valdes, J.E.N. (2022). Hermite-Hadamard-type Inequalities for Conformable Integrals. *Hacettepe Journal of Mathematics and Statistics*, *51*(3), 775-786.

Breckner, W.W. (1978). Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publ. Inst. Math.* (Beograd), *23*, 13-20.

Chen, J., Luo, C. (2023). Certain generalized Riemann–Liouville fractional integrals inequalities based on exponentially (h, m)-preinvexity. *Journal of Mathematical Analysis and Applications*, *2023*, 127731.

Dahmani, Z. (2010). On Minkowski and Hermite-Hadamard Integral Inequalities via Fractional Integration. *Annals of Functional Analysis*, *1*, 51-58.

Dragomir, S.S., Pearce, C.E.M. (2000). *Selected Topics on Hermite-Hadamard Inequalities and Applications*. RGMIA Monographs, Victoria University.

Eken, Z., Sezer, S., Tınaztepe, G. & Adilov, G. (2021). s-Convex functions in the fourth sense and some of their properties. *Konuralp Journal of Mathematics*. *9*(2), 260-267.

Eken, Z. (2021). Fejer inequality for s-convex functions in the fourth sense. *Journal of Mathematical Inequalities*, *16*(1), 19-34.

Eken Z. (2021). Hermite-Hadamard Type Inequalities Related to s-Convex Functions in the Third Sense. *Turkish Journal of Mathematics and Computer Science*, *13*(2), 318-330.

Fernandez, A., Mohammed, P. (2021). Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Mathematical Methods in the Applied Sciences*, *44* (10), 8414-8431.

Jleli, M., O'regan, D. & Samet, B. (2016). On Hermite-Hadamard type inequalities via generalized fractional integrals. *Turkish Journal of Mathematics*, *40*(6), 1221-1230.

Katugampola, U.N. (2014). A new approach to generalized fractional derivatives. *Bull. Math. Anal. App.*, *6*, 1-15.

Kemali, S. (2021). Hermite-Hadamard Type Inequality for s-Convex Functions in the Fourth Sense. *Turkish Journal of Mathematics and Computer Science*, *13*(2), 287-293.

Kemali, S., Sezer, S., Tınaztepe, G. & Adilov, G. (2021). *s*-convex functions in the third sense. *Korean Journal of Mathematics*, *29* (3), 593-602.

Kılıçman, A., Saleh, W. (2015). Notions of generalized s-convex functions on fractal sets. *Journal of Inequalities and Applications*, *2015*(1), 1-16.

Kilbas, A.A., Marichev, O.I. & Samko, S.G. (1993). *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, Switzerland.

Kwun, Y.C., Tanveer, M., Nazeer, W., Abbas, M. & Kang, S.M. (2019). Fractal generation in modified Jungck–S orbit. *IEEE Access*, *7*, 35060-35071.

Mohammed, P. (2021). Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. *Math Meth Appl Sci.*, *44*, 2314-2324.

Orlicz, W. (1961). A Note on Modular Spaces I. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys*, *9*(3), 157-162. https://www.fuw.edu.pl/\char126\relaxkostecki/scans/orlicz1961.pdf

Set, E., Sarikaya, M. Z., Ozdemir, M. E. & Yıldırım, H. (2014). The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results. *Journal of Applied Mathematics, Statistics and Informatics*, *10*(2), 69-83.

Sezer, S., Eken, Z., Tınaztepe, G. & Adilov, G. (2021). p-Convex functions and some of their properties. *Numerical Functional Analysis and Optimization*, *42*(4), 443-459.

Sezer, S. (2021a). Hermite-Hadamard Type Inequalities for the Functions Whose Absolute Values of First Derivatives are s-Convex. *Fundamental Journal of Mathematics and Applications*, *4*(2), 88-99.

Sezer, S. (2021b). The Hermite-Hadamard inequality for s-Convex functions in the third sense. *AIMS Mathematics*, *6* (7), 7719-7732.

Sun, H., Zhang, Y., Baleanu, D., Chen, W. & Chen, Y. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, *64*, 213-231.

Tınaztepe, G., Yesilce Isik, I., Kemali, S. & Adilov, G. (2021). Certain inequalities for s-convex functions via Caputo fractional derivative and Caputo-Fabrizio integral operator. *Journal of Mathematical Extension*, *15*(23), 1-18.

Tınaztepe, G., Sezer Evcan, S., Eken, Z., & Sezer, S. (2025). The Hermite-Hadamard type inequalities for the functions whose derivative is logarithmic p-convex function. *Hacettepe Journal of Mathematics and Statistics*, *54*(2), 404-413. https://doi.org/10.15672/hujms.1444589

Yang, Z. H., Zhang, W., & Chu, Y.M. (2016). Monotonicity and inequalities involving the incomplete gamma function. *Journal of Inequalities and Applications*, *2016*(1), 1-14. https://doi.org/10.1186/s13660-016-1160-7