arXiv:0710.5681v1 [math.NT] 30 Oct 2007

q-HARDY-BERNDT TYPE SUMS ASSOCIATED WITH q-GENOCCHI TYPE ZETA AND l-FUNCTIONS

YILMAZ SIMSEK

Abstract. The aim of this paper is to define new generating functions. By applying the Mellin transformation formula to these generating functions, we define q-analogue of Genocchi zeta function, q-analogue Hurwitz type Genocchi zeta function, q-analogue Genocchi type l-function and two-variable q-Genocchi type l-function. Furthermore, we construct new genereting functions of q-Hardy-Berndt type sums and q-Hardy-Berndt type sums attached to Dirichlet character. We also give some new relations related to q-Hardy-Berndt type sums and q-Genocchi zeta function as well.

1. Introduction

In [27], we defined generating functions. By using these functions, we constructed q-Riemann zeta function, q-L-function and q-Dedekind type sum. This sum is defined by means of the generating function, $Y_p(h,k;q)$:

$$Y_p(h,k,q)=\sum_{m=1}^{\infty}\frac{f(-\frac{2mi\pi h}{k},q)-f(\frac{2mi\pi h}{k},q)}{m^p}$$

where h and k are coprime positive integers and p is an odd integer ≥ 1, where

$$f(t,q)=\sum_{n=1}^{\infty}q^{-n}\exp\left(-q^{-n}[n]t\right), \text{ cf. ([23], [27]).} \tag{1.1}$$

In the remainder of our work, we use $\exp(x)=e^x$ and χ is a Dirichlet character of conductor $f\in\mathbb{Z}^+$, the set of positive integer numbers.

The q-Dedekind type sum is given by[27]:

Theorem 1. *Let h and k be positive integers and $(h,k)=1$ and assume that p is an odd integer ≥ 1. We have*

$$S_p(h,k;q)=\frac{p!}{(2\pi i)^p}Y_p(h,k;q).$$

By using (1.1), we construct q-analogous of Hardy-Berndt sums $s_1(h,k)$ and $s_4(h,k)$. Our aim is to define generating functions of q-analogous of the Hardy-Berndt type sums $S(h,k)$, $s_2(h,k)$, $s_3(h,k)$ and $s_5(h,k)$. We define

$$F(t,q)=\sum_{n=1}^{\infty}(-1)^n q^{-n}\exp\left(-q^{-n}[n]t\right). \tag{1.2}$$

1991 *Mathematics Subject Classification.* 11F20, 11B68, 11S40, 30B50, 44A05.

Key words and phrases. q-Genocchi zeta, q-L series, q-two-variable l-series, q-Dedekind sums, Hardy-Berndt sums.

Applying the Mellin transformations to the equation (1.2), we define, in Section 3, q-Genocchi type zeta functions, which implies the classical Genocchi zeta functions. By applying the Mellin transformation to

$$F(t,x,q)=F(t,q)\exp(-tx),$$

we define Hurwitz type q-Genocchi zeta functions.

Character analogues of (1.2) is defined by:

$$F_{\chi}(t,q)=\sum_{n=1}^{\infty}(-1)^{n}\chi(n)q^{-n}\exp\left(-q^{-n}[n]t\right). \tag{1.3}$$

By applying the Mellin transformations to the equation (1.3), we define q-Genocchi type l-function. By applying the Mellin transformation to

$$F_{\chi}(t,x,q)=F_{\chi}(t,q)\exp(-tx),$$

we define two-variable q-Genocchi type l-function. In Section 3, we prove the relations between q-Genocchi type l-function, two-variable q-Genocchi type l-function and Hurwitz type q-Genocchi zeta functions.

By using (1.1), (1.2) and (1.3), we prove our main results in Section 4-5. By using (1.2), $q$analogous of the Hardy-Berndt type sums' $S(h,k;q)$ is defined by means of the generating function, $Y_0(h,k;q)$:

$$Y_0(h,k;q)=\sum_{m=1}^{\infty}\frac{F(-\frac{(2m-1)i\pi h}{2k},q)-F(\frac{(2m-1)i\pi h}{2k},q)}{2m-1} \tag{1.4}$$

where h and k are coprime positive integers with $k\geq 1$.

Theorem 2. *Let h and k be denote relatively prime integers with $k\geq 1$. If $h+k$ is odd, then*

$$S(h,k;q)=\frac{4}{\pi i}Y_0(h,k;q). \tag{1.5}$$

Theorem 2 implies the classical Hardy-Berndt sums $S(h,k)$. The other theorems, which are related to q-Hardy-Bernd type sums, $s_j(h,k;q)$, $j=1,2,3,4,5$, are given in Section 4-5.

We define the sum $Y_{0,\chi}(t,q)$ as follows:

$$Y_{0,\chi}(h,k;q)=\sum_{m=1}^{\infty}\frac{F_{\chi}(-\frac{(2m-1)i\pi h}{2k},q)-F_{\chi}(\frac{(2m-1)i\pi h}{2k},q)}{2m-1}, \tag{1.6}$$

where h and k are coprime positive integers. Therefore, generalized q-Hardy-Berndt type sums $S_{\chi}(h,k;q)$, with attached to χ, are given by the following theorem:

Theorem 3. *Let h and k be denote relatively prime integers with $k\geq 1$. If $h+k$ is odd, then*

$$S_{\chi}(h,k;q)=\frac{4}{\pi i}Y_{0,\chi}(h,k;q). \tag{1.7}$$

2. Definition and Notations

The classical Dedekind sums $s(h,k)$ first arose in the transformation formulae of the logarithm of the Dedekind eta-function. Similarly, the Hardy-Berndt sums arose in the transformation formulae of the logarithm of the theta-functions, $Log\vartheta_n(0,q)$, $n=2,3,4$ cf. ([5], [7], [10], [11], [22]).

Due to Hardy[11] and Berndt[5], the Hardy-Berndt sums are defined by:

$$\begin{aligned}
S(h,k) &= \sum_{j=1}^{k-1}(-1)^{j+1+[\frac{hj}{k}]},\\
s_1(h,k) &= \sum_{j=1}^{k}(-1)^{[\frac{hj}{k}]}((\frac{j}{k})),\\
s_2(h,k) &= \sum_{j=1}^{k}(-1)^{j}((\frac{j}{k}))((\frac{hj}{k})),\\
s_3(h,k) &= \sum_{j=1}^{k}(-1)^{j}((\frac{hj}{k})),\\
s_4(h,k) &= \sum_{j=1}^{k-1}(-1)^{[\frac{hj}{k}]},\\
s_5(h,k) &= \sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k})).
\end{aligned}$$

Dieter[9] defined Hardy-Berndt sums by means of cotangent function. Goldberg[10] discovered some three-term and mixed three-term relations for Hardy-Berndt sums. His proofs are based on Berndt's transformation formulae for the logarithms of the classical theta-functions. For an elaboration of this connection and fundamental properties of Hardy-Berndt sums cf. (see for detail [3], [4], [6], [9], [18], [20], [24]).

Hardy-Berndt sums were represented as infinite trigonometric sums by Berndt and Goldberg[7] as follows:

Theorem 4. *Let h and k denote relatively prime integers with $k>0$. If $h+k$ is odd, then*

$$S(h,k)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\tan(\frac{\pi h(2n-1)}{2k})}{2n-1}, \tag{2.1}$$

if h is even and k is odd, then

$$s_1(h,k)=-\frac{2}{\pi}\sum_{\substack{n=1\\ 2n-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{\cot(\frac{\pi h(2n-1)}{2k})}{2n-1}, \tag{2.2}$$

if h is odd and k is even, then

$$s_2(h,k)=-\frac{1}{2\pi}\sum_{\substack{n=1\\ 2n\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{\tan(\frac{\pi hn)}{k})}{n}, \tag{2.3}$$

if k is odd, then

$$s_3(h,k)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\tan(\frac{\pi hn}{k})}{n}, \tag{2.4}$$

if h is odd, then

$$s_4(h,k)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cot(\frac{\pi h(2n-1)}{2k})}{2n-1}, \tag{2.5}$$

and if h and k are odd, then

$$s_5(h,k)=\frac{2}{\pi}\sum_{\substack{n=1\\ 2n-1\not\equiv 0(\mathrm{mod}\,k)}}^{\infty}\frac{\tan(\frac{\pi h(2n-1)}{2k})}{2n-1}. \tag{2.6}$$

Observe that if $q\to 1$, then Theorem 2 reduces to (2.1), which are stating in Section 4 in detail.

In [32], Sitaramachandrarao studied on Hardy-Berndt sums. He proved Reciprocity Law of these sums and gave the relations between Hardy-Berndt sums and Dedekind sums. In [28], we defined p-adic Hardy-type sums and p-adic q-higher-order Hardy-type sums. We gave p-adic continuous functions. By using these functions and p-adic q-integral, we obtain relations between p-adic q-higher-order Hardy-type sums, Bernoulli functions and Lambert series.

If $q\in\mathbb{C}$, the field of complex numbers, then we assume $\mid q\mid<1$. We set

$$[x]=[x:q]=\frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]=x$, cf. ([13], [23], [26]).

The Euler numbers E_n are usually defined by means of of the following generating function cf. ([16], [14], [19], [30]):

$$\frac{2}{\exp(t)+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\ |t|<\pi.$$

The Genocchi numbers G_n are usually defined by means of of the following generating function cf. ([16], [14], [17]):

$$\frac{2t}{\exp(t)+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\ |t|<\pi.$$

These numbers are classical and important in number theory. In [14], Kim defined generating function of the q-Genocchi numbers and q-Euler numbers as follows

$$[2]\exp(\frac{t}{1-q})\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+q^{n+1})(1-q)^n}\frac{t^n}{n!}=\sum_{m=0}^{\infty}E_{m,q}\frac{t^m}{m!},$$

where $E_{m,q}$ is denoted q-Euler numbers.

$$\begin{aligned}G_q(t) &= [2]t\sum_{m=0}^{\infty}(-1)^n q^n e^{[n]t}\\ &= \sum_{m=0}^{\infty}G_{m,q}\frac{t^m}{m!},\end{aligned}$$

where $G_{m,q}$ is denoted q-Genocchi numbers.

Genocchi zeta function is defined by cf. ([16], p. 108):

$$\zeta_G(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s},$$

where $s \in \mathbb{C}$.

In [17], by using q-Volkenborn Integral, Kim defined generating functions. By applying this generating function, they constructed q-Genocchi zeta function, the others q-function. They gave relations between q-Genocchi zeta function and q-Genocchi numbers. They also defined high order of q-Genocchi zeta function and q-Genocchi numbers and polynomials.

Kim and Rim[15] defined two-variable L-function. They gave main properties of this function. In [13], Kim constructed the two-variable p-adic q-L-function which interpolates the generalized q-Bernoulli polynomials attached to Dirichlet character. In [29], the author, Kim and Rim constructed the two-variable Dirichlet q-L-function and the two-variable multiple Dirichlet-type Changhee q-L-function.

We summarize our paper as follows:

In Section 3, we give new generating functions. By applying the Mellin transformation formula to these generating functions, we will define q-analogue of Genocchi zeta function, q-analogue Hurwitz type Genocchi zeta function, q-analogue l-function and two-variable q-l-function. In Section 4, by using generating functions in Section 3, we will construct new generating function which produce q-Hardy-Berndt type sums. In Section 5, by using generating functions in Section 3, we will construct new generating function attached to Dirichlet character which produces q-Hardy-Berndt type sums attached to Dirichlet character. In section 6, we define new generating functions. By applying Mellin transformation to these functions, we can give some new relations which are related to Riemann zeta functions, q-Riemann zeta function, q-L-function and q-Genocchi zeta function.

3. q-Genocchi zeta function and l-function

In [21], [23], and [26], the author defined generating functions, which are interpolates twisted Bernoulli numbers and polynomials, twisted Euler numbers and polynomials. In this section, we give new generating functions which produce q-Genocchi zeta functions and q-l-series with attached to Dirichlet character. Therefore, by using these generating functions, we construct new q-analogue of Hardy-Berndt sums in the next sections. We also give relations between these sums, q-Genocchi zeta functions and q-l-series.

Remark 1.

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

and

$$\zeta_G(s)\Gamma(s) = \int_0^{\infty} \frac{2x^{s-1}}{\exp(-x)+1} dx, \tag{3.1}$$

where $\Gamma(s)$ *is Euler's gamma function and*

$$\zeta_G(1-n) = -\frac{G_n}{n}, \ \ n > 1 \ \textit{cf. ([12], [16], p. 108, Eq. (2.43)).}$$

In (3.1), due to [16], [34] *and* [25],

$$\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}x^{s-1}\exp(-nx)dx,$$

and if $q\to 1$ *in (1.2), we have*

$$\begin{aligned}\lim_{q\to 1}2F(t,q)&=2\sum_{n=1}^{\infty}(-1)^n\exp(-nt)\\&=\frac{2}{\exp(t)+1}.\end{aligned}$$

According to the definition of Genocchi numbers:

$$\frac{2}{\exp(t)+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\ |t|<\pi$$

we get an asymptotic expansion near 0

$$\frac{1}{\exp(t)+1}\sim\sum_{k=-1}^{\infty}G_{k+1}\frac{t^k}{(k+1)!}$$

while

$$\frac{1}{\exp(t)+1}\sim 0$$

for t *near* ∞.

Applying the Mellin transformations to (1.2), we find that

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}F(t,q)dt\\&=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\sum_{n=1}^{\infty}(-1)^nq^{-n}\exp\left(-q^{-n}[n]t\right)\right)dt\\&=\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}(-1)^nq^{-n}\int_0^{\infty}t^{s-1}\exp\left(-q^{-n}[n]t\right)dt\\&=\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{(-1)^nq^{-n}}{\left(q^{-n}[n]\right)^s}\int_0^{\infty}u^{s-1}e^{-u}du\\&=\sum_{n=1}^{\infty}\frac{(-1)^nq^{-n}}{\left(q^{-n}[n]\right)^s}=\Im_q(s).\end{aligned}$$

The right-hand side of the above converges when $\mathrm{Re}(s)>1$. By using the above series, we are now ready to define q-analogue of the Genocchi zeta functions.

Definition 1. *Let* $s\in\mathbb{C}$ *and* $\mathrm{Re}(s)>1$. *q-analogue of the Genocchi type zeta function expressed by the formula*

$$\Im_{G,q}(s)=[2]\Im_q(s). \tag{3.2}$$

Remark 2. *Observe that when $q \to 1$, (3.2) reduces to ordinary Genocchi zeta functions (see* [16]*, p. 108). In* [8]*, Cenkci, Can and Kurt defined different type q-Genocch zeta functions, which is defined as follows*

$$\zeta_q^{(G)}(s) = q(1+q)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^n}{[n]^s}.$$

We define q-analogue of the Hurwitz type Genocchi zeta function by means of the generating function

$$F(t,x,q) = F(t,q)\exp(-tx) = \sum_{n=0}^{\infty}(-1)^n q^{-n}\exp\left(-\left(q^{-n}[n]+x\right)t\right). \tag{3.3}$$

By applying the Mellin transformations to (3.3), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F(t,x,q)dt\\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(\sum_{n=0}^{\infty}(-1)^n q^{-n}\exp\left(-\left(q^{-n}[n]+x\right)t\right)\right)dt\\
&= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}(-1)^n q^{-n}\int_0^{\infty} t^{s-1}\exp\left(-\left(q^{-n}[n]+x\right)t\right)dt\\
&= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-n}}{\left(q^{-n}[n]+x\right)^s}\int_0^{\infty} u^{s-1}e^{-u}du\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{-n}}{\left(q^{-n}[n]+x\right)^s} = \Im_q(s,x).
\end{aligned}$$

By using the above equation, we are ready to define q-analogue of the Hurwitz-type Genocchi zeta functions.

Definition 2. *Let $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$ and $0 < x \leq 1$. q-analogue of the Hurwitz-type Genocchi zeta function expressed by the formula*

$$\Im_{G,q}(s,x) := [2]\Im_q(s,x). \tag{3.4}$$

Remark 3. *We give another version of (3.3) as follows:*

$$\begin{aligned}
F(t,x,q) &= F(t,q)\exp(-t[x]) = \sum_{n=0}^{\infty}(-1)^n q^{-n}\exp\left(-\left(q^{-n}[n]+[x]\right)t\right)\\
&= \sum_{n=0}^{\infty}(-1)^n q^{-n}\exp\left(-q^{-n}t([n]+q^n[x]\right)
\end{aligned}$$

By using well-known the identity

$$[n+x] = [n]+q^n[x]$$

we have

$$F(t,x,q) = \sum_{n=0}^{\infty}(-1)^n q^{-n}\exp\left(-q^{-n}t[n+x]\right).$$

By applying the Mellin transformation to the above generating function, we obtain

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F(t,x,q)dt &= \sum_{n=0}^\infty \frac{(-1)^n q^{-n}}{\left(q^{-n}[n]+[x]\right)^s} \\ &= \sum_{n=0}^\infty \frac{(-1)^n q^{-n(1-s)}}{[n+x]^s} = \Im_q(s,x)\end{aligned}$$

Observe that if $x = 1$, *then* $\Im_{G,q}(s,x)$ *is reduced* $\Im_{G,q}(s)$ *and if* $q \to 1$, *then* $\Im_{G,q}(s,x) \to \Im_G(s,x)$. *A function* $\Im_G(s,x)$ *is called a ordinary Hurwitz-type Genocchi zeta function if* $\Im_G(s,x)$ *is expressed by the formula*

$$\Im(s,x) := 2\sum_{n=0}^\infty \frac{(-1)^n}{(n+x)^s},$$

where $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ *and* $0 < x \leq 1$ *(see* [17]*).*

By applying the Mellin transformations to the equation (1.3), we obtain

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F_\chi(t,q)dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\sum_{n=1}^\infty (-1)^n\chi(n)q^{-n}\exp\left(-q^{-n}[n]t\right)\right)dt \\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty (-1)^n\chi(n)q^{-n}\int_0^\infty t^{s-1}\exp\left(-q^{-n}[n]t\right)dt \\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty \frac{(-1)^n\chi(n)q^{-n}}{\left(q^{-n}[n]\right)^s}\int_0^\infty u^{s-1}e^{-u}du \\ &= \sum_{n=1}^\infty \frac{(-1)^n\chi(n)q^{-n}}{\left(q^{-n}[n]\right)^s} = l_q(s,\chi).\end{aligned}$$

Now, by using the above equation, we define q-analogue (Genocchi-type) l-function as follows:

Definition 3. *Let* χ *be a Dirichlet character. Let* $s \in \mathbb{C}$ *and* $\mathrm{Re}(s) > 1$. *(Genocchi-type) q-l-function expressed by the formula*

$$l_{G,q}(s,\chi) = [2]l_q(s,\chi). \tag{3.5}$$

A function $l_G(s,\chi)$ is called a ordinary Genocchi-type l-function if $l_G(s,\chi)$ is expressed by the formula

$$l(s,x) := 2\sum_{n=0}^\infty \frac{(-1)^n\chi(n)}{(n+x)^s},$$

where $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and $0 < x \leq 1$ [17].

Observe that when $\chi \equiv 1$, (3.5) reduces to (3.2):

$$l_q(s,1) = \Im_q(s).$$

In [8], Cenkci, Can and Kurt defined Genocch measure. By using this measure and Volkenborn Integral, they defined different type l-function. This functions interpolate generalized Genocchi numbers.

Relation between $\Im_{G,q}(s,x)$ and $l_{G,q}(s,\chi)$ and is given as follows:

Theorem 5. *Let χ be a Dirichlet character. We have*

$$l_{G,q}(s,\chi)=\frac{1}{[f]^s}\sum_{a=1}^{f}(-1)^a q^{a(s-1)}\chi(a)\Im_{G,q^f}\left(s,\frac{[a]}{[f]}\right). \tag{3.6}$$

Proof. Substituting $n=a+mf$, where $m=0,1,2,3,...,\infty$ and $a=1,...,f$ into (3.5), we obtain

$$l_{G,q}(s,\chi)=\sum_{a=1}^{f}(-1)^a q^{-a}\chi(a)\sum_{m=0}^{\infty}\frac{(-1)^{mf}q^{-mf}}{(q^{-a-mf}[a+mf])^s}.$$

By using $[a+mf]=[m:q^f][f]+q^{mf}[a]$ in the above equation, we obtain

$$\begin{aligned} l_{G,q}(s,\chi) &= \sum_{a=1}^{f}(-1)^a q^{-a}\chi(a)\sum_{m=0}^{\infty}\frac{(-1)^{mf}q^{-mf}}{\left(q^{-a-mf}\left([m:q^f][f]+q^{mf}[a]\right)\right)^s} \\ &= \frac{1}{[f]^s}\sum_{a=1}^{f}(-1)^a q^{a(s-1)}\chi(a)\sum_{m=0}^{\infty}\frac{(-1)^{mf}q^{-mf}}{\left(q^{-mf}[m:q^f]+\frac{[a]}{[f]}\right)^s} \end{aligned}$$

By using Eq.(3.4) in the above and after elementary calculations, we easily arrive at (3.6). □

Now, we define the following generating function

$$F_\chi(t,x,q)=F_\chi(t,q)e^{-tx}=\sum_{n=0}^{\infty}(-1)^n\chi(n)q^{-n}\exp\left(-(q^{-n}[n]+x)t\right). \tag{3.7}$$

By using the Mellin transformations in the above equation, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F_\chi(t,x,q)dt \\ &=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\sum_{n=0}^{\infty}(-1)^n\chi(n)q^{-n}\exp\left(-(q^{-n}[n]+x)t\right)\right)dt \\ &=\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}(-1)^n\chi(n)q^{-n}\int_0^\infty t^{s-1}\exp\left(-(q^{-n}[n]+x)t\right)dt \\ &=\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)q^{-n}}{\left(q^{-n}[n]+x\right)^s}\int_0^\infty u^{s-1}e^{-u}du \\ &=\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)q^{-n}}{\left(q^{-n}[n]+x\right)^s}=l_q(s,x,\chi). \end{aligned}$$

By using the above equation, we define the two-variable q-analogue l-function.

Definition 4.

$$l_{G,q}(s,x,\chi)=[2]l_q(s,x,\chi). \tag{3.8}$$

Relation between $\Im_{G,q}(s,x)$ and $l_{G,q}(s,x,\chi)$ and is given as follows

Theorem 6. *Let χ be a Dirichlet character of conductor f. We have*

$$l_{G,q}(s,x,\chi)=\frac{1}{[f]^s}\sum_{a=1}^{f}(-1)^a q^{a(s-1)}\chi(a)\Im_{G,q^f}\left(s,\frac{[a]+xq^a}{[f]}\right). \tag{3.9}$$

Proof. The proof of (3.9) follows precisely along the same lines as that of (3.6), and so we omit it. □

4. q-Hardy-Berndt type sums

In this section, we establish a general theorem related to the q-Hardy-Berndt type sums.

Theorem 7. *Let h and k be coprime positive integers with $h \geq 1$. We have*

$$Y_0(h,k;q) = 2i \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{-n} \sin(\frac{q^{-n}[n](2m-1)\pi h}{2k})}{2m-1}.$$

Proof. By using (1.2) and (1.4), we have

$$\begin{aligned} Y_0(h,k,q) &= \sum_{m=1}^{\infty} \frac{1}{2m-1} (\sum_{n=1}^{\infty} (-1)^n q^{-n} \exp(\frac{q^{-n}[n](2m-1)\pi hi}{2k}) \\ &\quad - \sum_{n=1}^{\infty} (-1)^n q^{-n} \exp(-\frac{q^{-n}[n](2m-1)\pi hi}{2k})) \\ &= \sum_{m=1}^{\infty} \frac{1}{2m-1} \sum_{n=1}^{\infty} (-1)^n q^{-n} (\exp(\frac{q^{-n}[n](2m-1)\pi hi}{2k}) \\ &\quad - \exp(-\frac{q^{-n}[n](2m-1)\pi hi}{2k})). \end{aligned}$$

Recalling that $2i \sin x = \exp(ix) - \exp(-ix)$, we easily complete the proof. □

We are now ready to prove the primary theorems of this section.

Proof of Theorem 2. Applying the generating function (1.4), and using (1.2) and Theorem 4, Eq. (2.1) and recalling the definition of $S(h,k)$, after some elementary calculations, we arrive at the desired result. □

Observe that when $q \to 1$ in Theorem 2, then we have

$$\begin{aligned} Y_0(h,k;1) &= \sum_{m=1}^{\infty} \frac{1}{2m-1} \sum_{n=1}^{\infty} (\exp(\frac{q^{-n}[n](2m-1)\pi hi}{2k}) \\ &\quad - \exp(-\frac{q^{-n}[n](2m-1)\pi hi}{2k})). \end{aligned}$$

Hence, applying the well-known series

$$\sum_{m=1}^{\infty} (-1)^n x^n = \frac{x}{x+1}$$

in the above equation, we deduce that

$$\begin{aligned} Y_0(h,k;1) &= \sum_{m=1}^{\infty} \frac{1}{2m-1} (\frac{\exp(\frac{\pi ih(2m-1)}{2k})}{1+\exp(\frac{\pi ih(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi ih(2m-1)}{2k})}{1+\exp(-\frac{\pi ih(2m-1)}{2k})}). \end{aligned} \tag{4.1}$$

We define

$$i\tan(x\pi) = \frac{\exp(2ix\pi)}{1+\exp(2ix\pi)} - \frac{\exp(-2ix\pi)}{1+\exp(-2ix\pi)}. \tag{4.2}$$

By (1.5), (4.1) and (4.2), we arrive at the following corollary.

Corollary 1. *Let h and k be denote relatively prime integers with $k \geq 1$. If $h+k$ is odd, then we have*

$$\begin{aligned} S(h,k;1) &= \frac{4}{\pi i} Y_0(h,k;1) \\ &= \frac{4}{\pi i}\sum_{m=1}^{\infty} \frac{1}{2m-1}\Big(\frac{\exp(\frac{\pi i h(2m-1)}{2k})}{1+\exp(\frac{\pi i h(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h(2m-1)}{2k})}{1+\exp(-\frac{\pi i h(2m-1)}{2k})}\Big). \end{aligned}$$

Remark 4. *The case $q \to 1$, $S(h,k,1)$ is denoted the classical Hardy-Berndt sums, which is given (2.1). Generalized Dedekind sums, $s(h,k;p)$ are expressible as infinite series related to certain Lambert series. A representation of $s(h,k;p)$ as infinite series and trigonometric representation was given by Apostol ([1], [2]). q-Dedekind type sums given by the author*[27].

We now define generating function of q-Hardy-Bernd type sums $s_j(h,k;q)$, $j = 1,2,3,4,5$. The definition of q-analogous of $s_j(h,k)$, $j = 1,2,3,4,5$ follow precisely along the same lines as the definition of the q-analogous of the Dedekind sums, which was established by the author[27].

q-Hardy-Bernd type sums $s_1(h,k;q)$ is defined by means of the generating function, $Y_1(h,k;q)$:

$$Y_1(h,k;q) = \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{f(-\frac{(2m-1)i\pi h}{2k},q) - f(\frac{(2m-1)i\pi h}{2k},q)}{2m-1}, \tag{4.3}$$

where $f(t,q)$ is defined by (1.1).

By using (4.3), (1.1) and (2.2), we obtain the fallowing theorem.

Theorem 8. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is even and k is odd, then*

$$s_1(h,k;q) = -\frac{2}{\pi i} Y_1(h,k;q).$$

Corollary 2. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is even and k is odd, then*

$$\begin{aligned} s_1(h,k;1) &= -\frac{2}{\pi i} Y_1(h,k;1) \\ &= -\frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{1}{2m-1}\Big(\frac{\exp(\frac{\pi i h(2m-1)}{2k})}{1-\exp(\frac{\pi i h(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h(2m-1)}{2k})}{1-\exp(-\frac{\pi i h(2m-1)}{2k})}\Big). \end{aligned}$$

Observe that the case $q \to 1$, $s_1(h,k;1)$ is the classical Hardy-Berndt sums, which is given (2.2).

By using (1.2), q-Hardy-Berndt type sums' $s_2(h,k;q)$ is defined by means of the generating function, $Y_2(h,k;q)$:

$$Y_2(h,k;q) = \sum_{\substack{m=1 \\ 2m \not\equiv 0 \bmod k}}^{\infty} \frac{F(-\frac{mi\pi h}{k},q) - F(\frac{mi\pi h}{k},q)}{m} \tag{4.4}$$

By using (4.4), (1.2) and (2.3), we obtain the fallowing theorem.

Theorem 9. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd and k is even, then*

$$s_2(h,k;q) = -\frac{1}{2\pi i} Y_2(h,k;q).$$

Corollary 3. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd and k is even, then*

$$\begin{aligned} s_2(h,k;1) &= -\frac{1}{2\pi i} Y_2(h,k;1) \\ &= -\frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m \not\equiv 0 \bmod k}}^{\infty} \frac{1}{m} \Big(\frac{\exp(\frac{\pi i h m}{k})}{1+\exp(\frac{\pi i h m}{k})} \\ &\quad - \frac{\exp(-\frac{\pi i h m}{k})}{1+\exp(-\frac{\pi i h m}{k})} \Big). \end{aligned}$$

Note that if $q \to 1$, then $s_2(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.3).

By using (1.2), q-Hardy-Berndt type sums' $s_3(h,k;q)$ is defined by means of the generating function, $Y_3(h,k;q)$:

$$Y_3(h,k;q) = \sum_{m=1}^{\infty} \frac{F(-\frac{mi\pi h}{k},q) - F(\frac{mi\pi h}{k},q)}{m} \tag{4.5}$$

By using (4.5), (1.2) and (2.4), we obtain the fallowing theorem.

Theorem 10. *Let h and k be denote relatively prime integers with $k \geq 1$. If k is odd, then*

$$s_3(h,k;q) = \frac{1}{\pi i} Y_3(h,k;q).$$

Corollary 4. *Let h and k be denote relatively prime integers with $k \geq 1$. If k is odd, then*

$$\begin{aligned} s_3(h,k;1) &= \frac{1}{\pi i} Y_3(h,k;1) \\ &= \frac{1}{\pi i} \sum_{m=1}^{\infty} \frac{1}{m} \Big(\frac{\exp(\frac{\pi i h m}{k})}{1+\exp(\frac{\pi i h m}{k})} \\ &\quad - \frac{\exp(-\frac{\pi i h m}{k})}{1+\exp(-\frac{\pi i h m}{k})} \Big). \end{aligned}$$

Observe that the case $q \to 1$, $s_3(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.4).

q-Hardy-Bernd type sums $s_4(h,k;q)$ is defined by means of the generating function, $Y_4(h,k;q)$:

$$Y_4(h,k;q) = \sum_{m=1}^{\infty} \frac{f(-\frac{(2m-1)i\pi h}{2k},q) - f(\frac{(2m-1)i\pi h}{2k},q)}{2m-1}, \tag{4.6}$$

where $f(t,q)$ is defined by (1.1).

By using (4.6), (1.1) and (2.5), we obtain the fallowing theorem.

Theorem 11. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd, then*

$$s_4(h,k;q) = \frac{4}{\pi i} Y_4(h,k;q).$$

Corollary 5. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd, then*

$$\begin{aligned} s_4(h,k;1) &= \frac{4}{\pi i} Y_4(h,k;1) \\ &= \frac{4}{\pi i} \sum_{m=1}^{\infty} \frac{1}{2m-1} \Big(\frac{\exp(\frac{\pi i h(2m-1)}{2k})}{1-\exp(\frac{\pi i h(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h(2m-1)}{2k})}{1-\exp(-\frac{\pi i h(2m-1)}{2k})} \Big). \end{aligned}$$

Observe that the case $q \to 1$, $s_4(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.5).

By using (1.2), q-Hardy-Berndt type sums' $s_5(h,k;q)$ is defined by means of the generating function, $Y_5(h,k;q)$:

$$Y_5(h,k;q) = \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{F(-\frac{(2m-1)i\pi h}{2k},q) - F(\frac{(2m-1)i\pi h}{2k},q)}{2m-1} \tag{4.7}$$

where h and k are coprime positive integers with $h \geq 1$.

By using (4.7), (1.2) and (2.6), we obtain the fallowing theorem.

Theorem 12. *Let h and k be denote relatively prime integers with $k \geq 1$. If h and k are odd, then*

$$s_5(h,k;q) = \frac{2}{\pi i} Y_5(h,k;q).$$

Corollary 6. *Let h and k be denote relatively prime integers with $k \geq 1$. If h and k are odd, then*

$$\begin{aligned} s_5(h,k;1) &= \frac{1}{\pi i} Y_5(h,k;1) \\ &= \frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{1}{m} \Big(\frac{\exp(\frac{\pi i h(2m-1)}{2k})}{1+\exp(\frac{\pi i h(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h(2m-1)}{2k})}{1+\exp(-\frac{\pi i h(2m-1)}{2k})} \Big). \end{aligned}$$

Observe that the case $q \to 1$, $s_5(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.6).

5. q-Hardy-Berndt type sums attached to Dirichlet character

In this section we define new generating functions which are generalizations of (1.4), (4.3), (4.4), (4.5), (4.6) and (4.7). By using (1.3) and

$$f_\chi(t,q) = \sum_{n=1}^{\infty} \chi(n) q^{-n} \exp(-q^n [n] t), \text{ cf. [27]} \tag{5.1}$$

we establish a general theorems related to q-Hardy-Berndt type sums with attached to Dirichlet character.

$$\begin{aligned} Y_{0,\chi}(h,k;q) &= \sum_{m=1}^{\infty} m^{-p} \sum_{n=1}^{\infty} \chi(n) \Big(\exp\Big(\frac{q^{-n}[n]\pi i h(2m-1)}{2k}\Big) \\ &\quad - \exp\Big(-\frac{q^{-n}[n]\pi i h(2m-1)}{2k}\Big)\Big) \\ &= 2i \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-p} \chi(n) q^{-n} \sin\Big(\frac{q^{-n}[n]\pi h(2m-1)}{2k}\Big). \end{aligned}$$

Thus we arrive at the following theorem:

Theorem 13. *Let h and k be denote relatively prime integers with $k \geq 1$. If $h+k$ is odd, then*

$$Y_{0,\chi}(h,k;q) = 2i \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-p} \chi(n) q^{-n} \sin\Big(\frac{q^{-n}[n]\pi h m}{2k}\Big).$$

Observe that

$$\lim_{q \to 1} Y_{0,\chi}(h,k;q) = 2i \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-p} \chi(n) \sin\Big(\frac{\pi h m n}{2k}\Big).$$

Proof of Theorem 3. The proof of this theorem is similar to that of Theorem 2. Applying the generating function to the equation (1.6), and using (1.3) and (2.1), we arrive at the desired result. □

Remark 5. *Observe that when $\chi \equiv 1$, principal character and and $q \to 1$, (1.7) reduces to (1.5) and $S_1(h,k;1)$ reduces to $S(h,k)$, which is given in (2.1). If $\chi \equiv 1$, principal character, then Theorem 3 and Theorem 13 reduce to Theorem 2 and Theorem 7, respectively.*

We now define generating function of the generalized q-Hardy-Bernd type sums $s_{j,\chi}(h,k;q)$, $j = 1,2,3,4,5$. The definition of q-analogous of $s_{j,\chi}(h,k)$, $j = 1,2,3,4,5$ follow precisely along the same lines as the definition of the q-Dedekind type sums, which was established by the author[27]. q-Hardy-Bernd type sums $s_{1,\chi}(h,k;q)$ is defined by means of the generating function, $Y_{1,\chi}(h,k;q)$:

$$Y_{1,\chi}(h,k;q) = \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{f_\chi(-\frac{(2m-1)i\pi h}{2k},q) - f_\chi(\frac{(2m-1)i\pi h}{2k},q)}{2m-1}, \tag{5.2}$$

where $f_\chi(t,q)$ is defined by (5.1).

By using (5.2), (5.1) and (2.2), we obtain the fallowing theorem.

Theorem 14. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is even and k is odd, then*

$$s_{1,\chi}(h,k;q) = -\frac{2}{\pi i} Y_{1,\chi}(h,k;q).$$

Corollary 7. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is even and k is odd, then*

$$\begin{aligned} s_{1,\chi}(h,k;1) &= -\frac{2}{\pi i} Y_{1,\chi}(h,k,1) \\ &= -\frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{2m-1} \Big(\frac{\exp(\frac{\pi i h(2m-1)n}{2k})}{1-\exp(\frac{\pi i h(2m-1)n}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h(2m-1)n}{2k})}{1-\exp(-\frac{\pi i h(2m-1)n}{2k})} \Big). \end{aligned}$$

Remark 6. *The case $\chi \equiv 1$, principal character and $q \to 1$, $s_{1,1}(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.2). If $\chi \equiv 1$, principal character, then Theorem 14 reduces to Theorem 8.*

By using (1.2), q-Hardy-Berndt type sums' $s_{2,\chi}(h,k;q)$ is defined by means of the generating function, $Y_{2,\chi}(h,k;q)$:

$$Y_{2,\chi}(h,k;q) = \sum_{\substack{m=1 \\ 2m \not\equiv 0 \bmod k}}^{\infty} \frac{F_\chi(-\frac{mi\pi h}{k},q) - F_\chi(\frac{mi\pi h}{k},q)}{m} \tag{5.3}$$

By using (5.3), (1.3) and (2.3), we obtain the fallowing theorem.

Theorem 15. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd and k is even, then*

$$s_{2,\chi}(h,k;q) = -\frac{1}{2\pi i} Y_{2,\chi}(h,k;q).$$

Corollary 8. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd and k is even, then*

$$\begin{aligned} s_{2,\chi}(h,k;1) &= -\frac{1}{2\pi i} Y_{2,\chi}(h,k,1) \\ &= -\frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m \not\equiv 0 \bmod k}}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{m} \Big(\frac{\exp(\frac{\pi i h m n}{k})}{1+\exp(\frac{\pi i h m n}{k})} \\ &\quad - \frac{\exp(-\frac{\pi i h m n}{k})}{1+\exp(-\frac{\pi i h m n}{k})} \Big). \end{aligned}$$

Remark 7. *The case$\chi \equiv 1$, principal character and $q \to 1$, $s_{2,1}(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.3). If $\chi \equiv 1$, principal character, then Theorem 15 reduces to Theorem 9.*

By using (1.3), q-Hardy-Berndt type sums' $s_{3,\chi}(h,k;q)$ is defined by means of the generating function, $Y_{3,\chi}(h,k;q)$:

$$Y_{3,\chi}(h,k;q) = \sum_{m=1}^{\infty} \frac{F_\chi(-\frac{mi\pi h}{k},q) - F_\chi(\frac{mi\pi h}{k},q)}{m} \tag{5.4}$$

By using (5.4), (1.3) and (2.4), we obtain the fallowing theorem.

Theorem 16. *Let h and k be denote relatively prime integers with $k \geq 1$. If k is odd, then*

$$s_{3,\chi}(h,k;q) = \frac{1}{\pi i} Y_{3,\chi}(h,k;q).$$

Corollary 9. *Let h and k be denote relatively prime integers with $k \geq 1$. If k is odd, then*

$$\begin{aligned} s_{3,\chi}(h,k;1) &= \frac{1}{\pi i} Y_{3,\chi}(h,k;1) \\ &= \frac{1}{\pi i} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{m} \Big(\frac{\exp(\frac{\pi i h m n}{k})}{1+\exp(\frac{\pi i h m n}{k})} \\ &\quad - \frac{\exp(-\frac{\pi i h m n}{k})}{1+\exp(-\frac{\pi i h m n}{k})} \Big). \end{aligned}$$

Remark 8. *The case $\chi \equiv 1$, principal character and $q \to 1$, $s_{3,1}(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.4). If $\chi \equiv 1$, principal character, then Theorem 16 reduces to Theorem 10.*

q-Hardy-Bernd type sums $s_{4,\chi}(h,k;q)$ is defined by means of the generating function, $Y_{4,\chi}(h,k,q)$:

$$Y_{4,\chi}(h,k;q) = \sum_{m=1}^{\infty} \frac{f_\chi(-\frac{(2m-1)i\pi h}{2k},q) - f_\chi(\frac{(2m-1)i\pi h}{2k},q)}{2m-1}, \tag{5.5}$$

where $f_\chi(t,q)$ is defined by (5.1).

By using (5.5), (5.1) and (2.5), we obtain the fallowing theorem.

Theorem 17. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd, then*

$$s_{4,\chi}(h,k;q) = \frac{4}{\pi i} Y_{4,\chi}(h,k;q).$$

Corollary 10. *Let h and k be denote relatively prime integers with $k \geq 1$. If h is odd, then*

$$\begin{aligned} s_{4,\chi}(h,k;1) &= \frac{4}{\pi i} Y_{4,\chi}(h,k;1) \\ &= \frac{4}{\pi i} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{2m-1} \Big(\frac{\exp(\frac{\pi i h n(2m-1)}{2k})}{1-\exp(\frac{\pi i h n(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h n(2m-1)}{2k})}{1-\exp(-\frac{\pi i h n(2m-1)}{2k})} \Big). \end{aligned}$$

Remark 9. *The case $\chi \equiv 1$, principal character and $q \to 1$, $s_{4,1}(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.5). If $\chi \equiv 1$, principal character, then Theorem 17 reduces to Theorem 11.*

By using (1.3), q-Hardy-Berndt type sums' $s_{5,\chi}(h,k;q)$ is defined by means of the generating function, $Y_{5,\chi}(h,k;q)$:

$$Y_{5,\chi}(h,k;q) = \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \frac{F_\chi(-\frac{(2m-1)i\pi h}{2k}, q) - F_\chi(\frac{(2m-1)i\pi h}{2k}, q)}{2m-1} \tag{5.6}$$

where h and k are coprime positive integers with $h \geq 1$.

By using (5.6), (1.3) and (2.6), we obtain the fallowing theorem.

Theorem 18. *Let h and k be denote relatively prime integers with $k \geq 1$. If h and k are odd, then*

$$s_{5,\chi}(h,k;q) = \frac{2}{\pi i} Y_{5,\chi}(h,k,q).$$

Corollary 11. *Let h and k be denote relatively prime integers with $k \geq 1$. If h and k are odd, then*

$$\begin{aligned} s_{5,\chi}(h,k;1) &= \frac{1}{\pi i} Y_{5,\chi}(h,k,1) \\ &= \frac{2}{\pi i} \sum_{\substack{m=1 \\ 2m-1 \not\equiv 0 \bmod k}}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{2m-1} \Big(\frac{\exp(\frac{\pi i h n(2m-1)}{2k})}{1+\exp(\frac{\pi i h n(2m-1)}{2k})} \\ &\quad - \frac{\exp(-\frac{\pi i h n(2m-1)}{2k})}{1+\exp(-\frac{\pi i h n(2m-1)}{2k})} \Big). \end{aligned}$$

Remark 10. *The case $\chi \equiv 1$, principal character and $q \to 1$, $s_{5,1}(h,k;1)$ is denoted the classical Hardy-Berndt sums, which is given (2.6). If $\chi \equiv 1$, principal character, then Theorem 18 reduces to Theorem 12.*

6. New generating functions related to the Riemann zeta function, L-function and Genocchi zeta function

In this section, our goal is to define new generating functions. By applying the Mellin transformation to these functions, we can give some new relations which are related to Riemann zeta functions, q-Riemann zeta function, q-L-function and q-Genocchi zeta function.

We define

$$y_0(t,q)=\sum_{m=1}^{\infty}\frac{F(-(2m-1)it,q)-F((2m-1)it,q)}{2m-1}, \tag{6.1}$$

where $F(t,q)$ is defined by the relation (1.2).

Theorem 19. *We have*

$$\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_0(t,q)dt=(i)^{-s}((-1)^{-s}-1)\Im_{G,q}(s)\zeta^*(s+1),$$

where $\Im_{G,q}(s)$ is the Genocchi zeta functions and

$$\zeta^*(s):=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^s}.$$

Proof. By applying the Mellin transformation to the equation (6.1) and using (1.2), we find that

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_0(t,q)dt\\
&=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\sum_{m=1}^{\infty}\frac{F(-(2m-1)it,q)-F((2m-1)it,q)}{2m-1}\right)dt\\
&=\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{1}{2m-1}(\sum_{n=1}^{\infty}(-1)^nq^{-n}\int_0^{\infty}t^{s-1}\exp\left(q^{-n}[n](2m-1)ti\right)dt\\
&-\sum_{n=1}^{\infty}(-1)^nq^{-n}\int_0^{\infty}t^{s-1}\exp\left(-q^{-n}[n](2m-1)ti\right)dt)\\
&=\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{(i)^{-s}((-1)^{-s}-1)}{(2m-1)^{s+1}}\left(\sum_{n=1}^{\infty}\frac{(-1)^nq^{-n}}{(q^{-n}[n])^s}\int_0^{\infty}u^{s-1}e^{-u}du\right)\\
&=(i)^{-s}((-1)^{-s}-1)\sum_{m=1}^{\infty}\frac{1}{(2m-1)^{s+1}}\sum_{n=1}^{\infty}\frac{(-1)^nq^{-n}}{(q^{-n}[n])^s}.
\end{aligned}$$

After some elementary calculations, we obtain the desired result. □

Remark 11. *In* [31]*, p. 96, Eq. 2.3 (1), Srivastava and Choi defined Riemann zeta function as follows:*

$$\zeta(s)=\begin{cases}\sum_{m=1}^{\infty}n^{-s}=\frac{1}{1-2^{-s}}\sum_{m=1}^{\infty}(2m-1)^{-s}, & \mathrm{Re}(s)>1\\ (1-2^{1-s})^{-1}\sum_{m=1}^{\infty}(-1)^{m-1}m^{-s}, & (\mathrm{Re}(s)>0,\ s\neq 1.\end{cases}$$

For $\mathrm{Re}(s)>1$*, by using this definition, relation between $\zeta^*(s)$ and $\zeta(s)$ is given by*

$$\zeta^*(s)=(1-2^{-s})\zeta(s).$$

In [31]*, p. 121, Eq. 2.5 (1), They defined Hurwitz-Lerch zeta functions as follows*

$$\Phi(z,s,a) := \sum_{m=1}^{\infty} \frac{z^m}{(m+a)^s},$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$*;* $s \in \mathbb{C}$ *when* $\mid z \mid < 1$*;* $\mathrm{Re}(s) > 1$ *when* $\mid z \mid = 1$*. In* [31]*, p. 339, Eq. 6.1 (18), They gave the following identity*

$$\sum_{m=1}^{\infty} \frac{z^m}{(2m-1)^s} = (2b)^{-s} \sum_{j=1}^{b} \Phi(z^b, s, \frac{2j-1}{2b}) z^{j-1}.$$

By substituting $\mathrm{Re}(s) > 1$ *when* $z = 1$ *into the above identity, we have*

$$\begin{aligned} \zeta^*(s) &= (2b)^{-s} \sum_{j=1}^{b} \Phi(1, s, \frac{2j-1}{2b}) \\ &= (2b)^{-s} \zeta(s, \frac{2j-1}{2b}), \end{aligned}$$

where $\zeta(s,x)$ *denotes Hurwitz zeta functions.*

We define the following generating function:

$$y_1(t,q) = \sum_{m=1}^{\infty} \frac{f(-(2m-1)it, q) - f((2m-1)it, q)}{2m-1}, \tag{6.2}$$

where $f(t,q)$ is defined by (1.1).

Theorem 20. *We have*

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} y_1(t,q) dt = (i)^{-s}((-1)^{-s} - 1)\zeta_q(s)\zeta^*(s+1),$$

where $\zeta_q(s)$ *is the q-Riemann zeta function, which is defined in* [27].

Proof. By applying the Mellin transformation to the equation (6.2) and using (1.1), we find that

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} y_1(t,q) dt \\ = &\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{m=1}^{\infty} \frac{f(-(2m-1)it, q) - f((2m-1)it, q)}{2m-1} \right) dt. \end{aligned}$$

The remainder of the proof runs parallel to that of Theorem 19, so we choose to omit the details involved. □

We define the following generating function:

$$y_2(t,q) = \sum_{m=1}^{\infty} \frac{F(-tmi, q) - F(tmi, q)}{m}.$$

By applying the Mellin transformation to the above function, we arrive at the following theorem:

Theorem 21. *We have*

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} y_2(t,q) dt = (i)^{-s}((-1)^{-s} - 1)\Im_{G,q}(s)\zeta(s+1),$$

where $\zeta(s+1)$ *is the Riemann zeta functions.*

Remark 12. *Proof of Theorem 21 run parallel to that of Theorem 19, so we choose to omit the details involved.*

We define the following generating function attached to Drichlet character:

$$y_{0,\chi}(t,q)=\sum_{m=1}^{\infty}\frac{F_{\chi}(-(2m-1)it,q)-F_{\chi}((2m-1)it,q)}{2m-1}, \tag{6.3}$$

where $F_{\chi}(t,q)$ is defined by (5.1).

Theorem 22. *We have*

$$\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_{0,\chi}(t,q)dt=(i)^{-s}((-1)^{-s}-1)l_{G,q}(s,\chi)\zeta^{*}(s+1).$$

Proof. By applying the Mellin transformation to the equation (6.3) and using (5.1), we find that

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_{0,\chi}(t,q)dt\\
&=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\sum_{m=1}^{\infty}\frac{F_{\chi}(-(2m-1)it,q)-F_{\chi}((2m-1)it,q)}{2m-1}\right)dt\\
&=\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{1}{2m-1}(\sum_{n=1}^{\infty}(-1)^{n}\chi(n)q^{-n}\int_0^{\infty}t^{s-1}\exp\left(q^{-n}[n](2m-1)ti\right)dt\\
&-\sum_{n=1}^{\infty}(-1)^{n}\chi(n)q^{-n}\int_0^{\infty}t^{s-1}\exp\left(-q^{-n}[n](2m-1)ti\right)dt)\\
&=\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{(i)^{-s}((-1)^{-s}-1)}{(2m-1)^{s+1}}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n}\chi(n)q^{-n}}{(q^{-n}[n])^{s}}\int_0^{\infty}u^{s-1}e^{-u}du\right)\\
&=(i)^{-s}((-1)^{-s}-1)\sum_{m=1}^{\infty}\frac{1}{(2m-1)^{s+1}}\sum_{n=1}^{\infty}\frac{(-1)^{n}\chi(n)q^{-n}}{(q^{-n}[n])^{s}}.
\end{aligned}$$

After some elementary calculations, we obtain the desired result. □

We define the following generating function:

$$y_{1,\chi}(t,q)=\sum_{m=1}^{\infty}\frac{f_{\chi}(-(2m-1)it,q)-f_{\chi}((2m-1)it,q)}{2m-1}, \tag{6.4}$$

where $f_{\chi}(t,q)$ is defined by (5.1).

Theorem 23. *We have*

$$\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_{1,\chi}(t,q)dt=(i)^{-s}((-1)^{-s}-1)L_{q}(s,\chi)\zeta^{*}(s+1),$$

where $L_q(s,\chi)$ is the q-L function, which is defined in [27].

Proof. By applying the Mellin transformation to the equation (6.4) and using (5.1), we find that

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}y_{1,\chi}(t,q)dt\\
=\;&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\sum_{m=1}^{\infty}\frac{f_{\chi}(-(2m-1)it,q)-f_{\chi}((2m-1)it,q)}{2m-1}\right)dt.
\end{aligned}$$

The remainder of the proof runs parallel to that of Theorem 22, so we choose to omit the details involved. □

Remark 13. *By setting $t = \frac{\pi i h}{k}$, h, $k \in \mathbb{Z}$ with $(h,k) = 1$ in (6.1), (6.2), (6.3) and (6.4), we obtain generating functions in Section 4 and 5, respectively.*

Acknowledgement 1. *This paper was supported by the Scientific Research Project Administration Akdeniz University.*

References

[1] T. M. Apostol, Generalized Dedekind sums an transformation formulae of certain Lambert series, Duke Math. J., 17 (1950), 147-157.
[2] T. M. Apostol, Theorems on generalized Dedekind sums, Pacific J. Math., 2 (1952), 1-9.
[3] T. M. Apostol and T. H. Vu, Elementary proofs of Berndt's reciprocity laws, Pasific J. Math., 98 (1982), 17-23.
[4] B. C. Berndt, Dedekind sum and paper of G. H. Hardy, J. London Math. Soc., 13 (1976), 129-136.
[5] B. C. Berndt, Analytic Eisenstein series, theta-functions and series relations in the spirit of Ramanujan, J. Reine Angew. Math., 303/304 (1978), 332-365.
[6] B. C. Berndt and U. Dieter, Sums involving the greatest integer function and Riemann Stieltjes integration, J. Reine Angew. Math., 337 (1982), 208-220.
[7] B. C. Berndt and L. A. Goldberg, Analytic properties of arithmetic sums Arasing in the theory of the classical theta-functions, Siam. J. Math. Anal., 15 (1984), 143-150.
[8] M. Cenkci, M. Can and V. Kurt, q-extensions of Genocchi numbers, J. Korean Math. Soc., 43(1) (2006), 183-198.
[9] U. Dieter, Cotangent sums a further generalization of Dedekind sums, J. Number Theory, 18 (1984), 289-305.
[10] L. A. Goldberg, Transformation of theta-functions and analogues of Dedekind sums, Thesis, University of Illinois Urbana, 1981.
[11] G. H. Hardy, On certain series of discontinous functions connected with the Modular Functions, Quart. J. Math., 36 (1905), 93-123 = Collected Papers, Vol.IV, 362-392. Clarendon Press, Oxford 1969.
[12] L.-C. Jang, T. Kim, D.- H Lee, and AD.-W. Park, An application of polylogarithms in the analogs of Genocchi numbers, NNTDM, 7 (3) (2001), 65-69.
[13] T. Kim, Power series and asymptotic series associated with the q-analogue of two-variable p-adic L-function, Russ. J. Math Phys., 12 (2) (2005), 186-196.
[14] T. Kim, q-Euler and Genocchi numbers, arXiv:math.NT/0506278 v1, 14 June 2005.
[15] T. Kim and S. -H. Rim, A note on two variable Dirichlet L-function, Adv. Stud. Contep. Math., 10 (2005), 1-7.
[16] T. Kim et al, Introduction to Non-Archimedian Analysis, Kyo Woo Sa (Korea), 2004. (http://www.kyowoo.co.kr).
[17] T. Kim et al, q-Euler numbers q-Genocchi numbers and polynomials of high order related to Genocchi zeta function, preprint.
[18] M. R. Pettet and R. Sitaramachandraro, Three-term relations for Hardy sums, J. Number Theory, 25 (1989), 328-339.
[19] K. Shiratani and S. Yamamoto, On a p-adic interpolation function for the Euler numbers and its derivative, Mem. Fac. Kyushu Uni., 39 (1985), 113-125.
[20] Y. Simsek, Theorems on three-term relations for Hardy sums, Turkish J.of Math., 22 (1998), 153-162.
[21] Y. Simsek, O. Yurekli and V. Kurt, On interpolation functions of the twisted generalized Frobenius-Euler numbers, Adv. Stud. Contemp. Math., 15(2) (2007), 187-194.
[22] Y. Simsek, Relation between theta-function Hardy sums Eisenstein and Lambert series in the transformation formula of $\log \eta_{g,h}(z)$, J. Number Theory, 99(2) (2003), 338-360.
[23] Y. Simsek, On q-analogue of the twisted L-function and q-twisted Bernoulli numbers, J. Korean Math. Soc. No: 6, 40 (2003), 963-975.
[24] Y. Simsek, On generalized Hardy sums $S_5(h,k)$, Ukrainian Math. J., 56(10) (2004), 1434-1440.

[25] Y. Simsek, S. Yang, Transformation of four Titchmarsh-type infinite integrals and generalized Dedekind sums associated with Lambert series. Adv. Stud. Contemp. Math., 9(2) (2004), 195–202.
[26] Y. Simsek, q-analogue of the twisted l-series and q-twisted Euler numbers, J. Number Theory, 110(2) (2005), 267-278.
[27] Y. Simsek, q-Dedekind type sums related to q-zeta function and basic L-series, to appear in Journal of the Mathematical Analysis and Applications.
[28] Y. Simsek, p-adic q-higher-order Hardy-type sums, J. Korean Math. Soc., 43(1) (2006), 111-131.
[29] Y. Simsek, D. Kim and S.-H. Rim, On the two-variable Dirichlet q-L-series, Adv. Stud. Contep. Math., 10(2) (2005).
[30] H. M. Srivastava, T. Kim and Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series, Russ. J. Math Phys., 12(2) (2005), 241-268.
[31] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Acedemic Publishers, Dordrecht, Boston and London, 2001.
[32] R. Sitaramachandrarao, Dedekind and Hardy sums, Acta Arith., XLVIII (1978), 325-340.
[33] H. Tsumura, A note on q-analogue of the Dirichlet series and q-Bernoulli numbers, J. Number Theory, 39 (1991), 251-256.
[34] M. Waldschmidt, P. Moussa, J. M. Luck, C. Itzykson, From Number Theory to Physics, Springer-Verlag, 1995.

AKDENIZ UNIVERSITY FACULTY OF ART AND SCIENCE DEPARTMENT OF MATHEMATICS 07058, ANTALYA, TURKEY, Tel: ++90 242 310 23 43, Fax:+ +90242227 89 11
E-mail address: yilmazsimsek@hotmail.com, simsekyil63@yahoo.com