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Atomic Size-Dependent Nonlocal Torsional Vibration of Circular Rod

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Abstract

This paper presents a comprehensive analysis of the torsional vibration of circular rods using nonlocal elasticity theory, emphasizing its applicability in nanoscale systems. The equation of motion for nonlocal torsional vibration of nanoscale rods is first derived. After applying some transformations to the obtained equation of motion, the frequency equation is formulated. In addition, a nonlocal finite element model (NL-FEM) for the rod is developed using the weighted residual method.

Keywords: Torsional vibration, Nonlocal elasticity, Nanoscale structures, Finite element method

1. Introduction

Nano-sized systems have revolutionized various fields by driving significant advancements in material design and development. These systems enable the manipulation of material properties at the atomic and molecular scale, offering performance and functionality previously unattainable. Materials at this size exhibit different behavior from macro-sized material properties. Dynamic properties such as torsional vibrations become more pronounced in nanostructures. Therefore, understanding the mechanical behavior of materials at the nano-sized is of great importance, as the deformations caused by vibrations can have significant impacts on the design and performance of nanomaterials.

Nano-sized structures, such as nanoparticles, nanowires, nanorods, nanobeams, nanorings, nanosheets, nanoplates, and nanotubes (CNT), combined to form nanomachines [1]. The products developed through nanotechnology find extensive applications across a wide range of fields, including medicine, materials science, aerospace, defence, electronics, and computer technology. However, the design of nano-sized structures requires a thorough understanding of their mechanical behavior. Numerous researchers have observed significant changes in the mechanical properties of materials as they transition from the macro scale to the nano scale. To illustrate the changes in mechanical properties of certain materials at the nanoscale, the colour of gold darkens and its melting point decreases; copper becomes a semiconductor; silica glass exhibits flexible behavior; furthermore, carbon nanotubes and graphene at the nanoscale can possess significantly higher strength than steel. It has been observed by experimental studies that the mechanical behavior of such materials cannot be carried out in the framework of rules of classical physics. Theoretical analyses performed to understand the mechanical behavior at the nanoscale have reached an important position in scientific literature in the last two decades. Here, various approaches such as nonlocal elasticity, couple stress, strain gradient, surface energy and doublet mechanics constitute the class of theoretical analysis [2]. In this study, the nonlocal elasticity theory to be used for the solution accepts the idea that the relationships

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between stress and deformation in a material are influenced not only by the state at a given point but also by the states of surrounding points, in contrast to classical elasticity theory. This is particularly important in cases involving very thin structures, nanostructures, or large deformation. Some of the studies in the literature on nonlocal theory are summarized below.

In the past years, Eringen and Edelen [3-8] applied the theory of nonlocal elasticity to various problems and made important contributions to science by obtaining equations. Subsequently, another important study on this topic was conducted by Reddy [9]. Reddy conducted bending, vibration, and buckling analyses of simply supported beams using the nonlocal Euler-Bernoulli beam theory, Timoshenko beam theory, his own developed theory, and Levinson beam theory. Aydoğdu [10] performed the axial free vibration analysis of nanoscale rods based on the nonlocal elasticity theory. Civalek and Demir [11] conducted a bending analysis of microtubules using Euler-Bernoulli beam theory and nonlocal elasticity theory to account for dimensional effects. Civalek et al. [12] studied the bending of nonlocal beams with differential quadrature solution. Civalek and Demir [13] presented the differential quadrature solution for bending and buckling of cantilever nanobeam. Li [14] carried out a torsional vibration analysis of carbon nanotubes and presented these analyses based on two different nonlocal models and a semi-continuum model. Xu *et al.* [15] conducted a free vibration analysis of nanorods in their study using a combination of nonlocal elasticity and modified strain gradient elasticity theories. Besides these studies, the use of finite element formulation in nonlocal mechanical analyses of nano-sized structures has become increasingly widespread in recent years. The free vibration analysis of nanoscale axial bars with a single crack using nonlocal finite element formulation was investigated by Numanoğlu *et al.* [16]. Fakher and Hosseini-Hashemi [17] developed an efficient finite element formulation to prevent shear locking in the vibration analysis of Timoshenko nanobeams based on the two-phase nonlocal model.

This study presents a finite element solution for the torsional vibration of circular cross-section rods based on nonlocal elasticity theory. The first section explains the concepts using nanoscale examples and provides an overview of studies related to nonlocal elasticity theory that accounts for size effects. In the second section, the formulation of this theory is explained. The third section derives the governing equation for the torsional vibration of circular rods. The fourth section presents the nonlocal finite element formulation. On the other hand, vibration and stability analyses of elastic solids under different effects have been investigated using various numerical techniques [18-24].

2. Nonlocal Elasticity Theory (NL)

In micro and nanoscale structures where the size effect is important, various theories are used since the classical elasticity theory is insufficient. Among these theories, especially the nonlocal elasticity theory proposed by Eringen [25] is widely preferred. In this theory, the long-range force about the atoms is considered, and thus an internal scale effect is introduced in the constitutive equation [26]. Accordingly, the nonlocal Cauchy equation of motion of a homogeneous and isotropic elastic solid:

$$\sigma_{kl,k} + \rho\left(f_l - \frac{\partial^2 u_l}{\partial t^2}\right) = 0 \tag{1}$$

$$\sigma_{kl}(x) = \int_V \alpha(|x' - x|, \chi)\tau_{kl}(x')dV(x'), \tag{2}$$

$$\tau_{kl}(x') = \lambda\varepsilon_{rr}(x')\delta_{kl} + 2\mu\varepsilon_{kl}(x'), \tag{3}$$

$$\varepsilon_{kl}(x') = \frac{1}{2}\left(\frac{\partial u_k(x')}{\partial x_l'} + \frac{\partial u_l(x')}{\partial x_k'}\right) \tag{4}$$

In these equations σ_{kl} is the nonlocal stress tensor, ρ is the mass density of the body, f_l is the mass or acting force, u_l is the displacement vector of the body for any reference point x, t is time, $\alpha|x - x'|$ is the distance in Euclidean form $\tau_{kl}(x')$ is the classical (Cauchy) or local stress tensor of the body at point x', V is the volume occupied by the elastic body, $\varepsilon_{kl}(x')$ is the linear strain tensor of the body at point x', λ and μ are Lame constants. The nonlocal kernel $\alpha|x - x'|$ defines the strain effect of the stress value of the elastic body at point x at point x'. Again, χ in the equations depends on the ratio $(\frac{e_0 a}{l})$ and is a material constant. Here, a is the internal (granular) distance, and l external characteristic length (refraction or wavelength) of the material and e_0 is the coefficient experimentally determined and proposed for each material.

It is stated that it is very difficult to analyze elasticity problems with the integral constitutive relation given in Eq. (2). As a result, an equivalent differential form of the expression given in Eq. (2) was developed to analyze different problems [27]. Moreover, the differential form, which provides a simpler solution compared to the integral form, is widely used in nanostructures [28]. Stress relation at the constitutive equation of nonlocal elasticity theory was proposed by Eringen [5] as follows:

$$[1 - (e_0 a)^2\nabla^2]\sigma_{kl} = \tau_{kl} \tag{5}$$

where ∇^2 is the Laplacian.

3. Torsional Vibration of Circular Rod

The relationship between the amount of rotation $\theta(x,t)$ at time t and the torsional moment $M_T(x,t)$ at a cross-section located at any distance x along the rod is defined by the following equation.

$$M_T(x,t) = GI_p\frac{\partial\theta(x,t)}{\partial x} \tag{6}$$

where $G = G(x)$ is the shear modulus of material and $I_p = I_p(x)$ is the polar moment of inertia of the cross-sectional area at x with respect to the center. The free-body diagram of a small portion of the torsion rod between x and $x + \Delta x$ is shown in Figure 1.

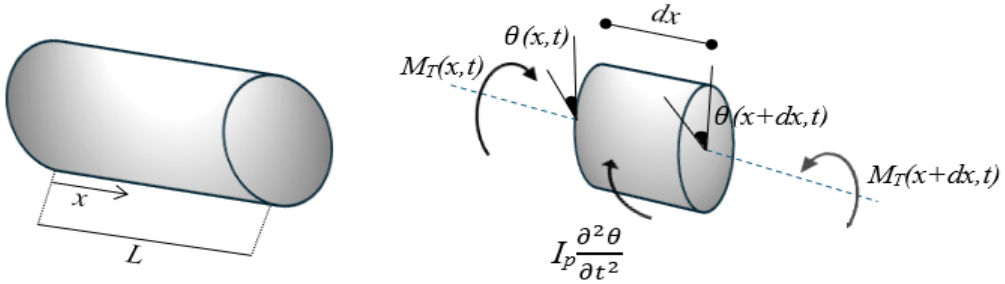


Fig. 1. Torsional vibration of circular rod

The moment of mass inertia of the part at unit length around the axis of rotation is $\rho I_p dx$ and the acceleration reaction moment acting on this part is expressed as follows.

$$\sum M_T(x,t) = \rho I_p \frac{\partial^2 \theta(x,t)}{\partial t^2} dx \tag{7}$$

Equilibrium of all moments acting on the part

$$M_T(x,t) - \left[M_T(x,t) + \frac{\partial M_T(x,t)}{\partial x} dx \right] = \rho I_p \frac{\partial^2 \theta(x,t)}{\partial t^2} dx \tag{8}$$

When the equation is simplified, the following relation is obtained

$$G \frac{\partial^2 \theta}{\partial x^2} = \rho \frac{\partial^2 \theta}{\partial t^2} \tag{9}$$

In the theory of nonlocal elasticity, the relationship between stress and strain, as well as between torsional moments and displacement, can be expressed as follows [9, 29]

$$\tau - (e_0 a)^2 \frac{\partial^2 \tau}{\partial x^2} = 2G\varepsilon \tag{10}$$

$$M_T - (e_0 a)^2 \frac{\partial^2 M_T}{\partial x^2} = G I_p \frac{\partial \theta}{\partial x} \tag{11}$$

If both sides of Eq. (11) are differentiated once with respect to position and Eq. (9) is substituted in the new equation obtained [30]

$$G I_p \frac{\partial^2 \theta}{\partial x^2} + (e_0 a)^2 \rho I_p \frac{\partial^4 \theta}{\partial x^2 \partial t^2} - \rho I_p \frac{\partial^2 \theta}{\partial t^2} = 0 \tag{12}$$

The equation above is the governing equation for the nonlocal torsional vibration of nanoscale rods. When the value of the nonlocal parameter $(e_0 a)$ is set to zero, it reduces to the classical rod model.

3.1. Continuous System

The following transformation will be applied for the nonlocal axial vibration of nanorods.

$$\theta(x,t) = \varphi(x) \sin(\omega t - \alpha) \tag{13}$$

where φ is modal function, ω is vibration frequency and α is phase angle. Eq. (12) turns into Eq. (16) by applying the Eq. (13)

$$G I_p \frac{\partial^2 \varphi}{\partial x^2} + (e_0 a)^2 \left(-\rho I_p \frac{\partial^2 \varphi}{\partial x^2} \omega^2 \right) + \rho I_p \omega^2 \varphi = 0 \tag{14}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \lambda^2 \varphi = 0 \tag{15}$$

$$\lambda^2 = \frac{\rho I_p \omega^2}{GI_p - (e_0 a)^2 \rho I_p \omega^2} \tag{16}$$

$\varphi = e^{mx}$ transformation is applied to Eq. (14)

$$m^2 e^{mx} + \lambda^2 e^{mx} = 0 \tag{17}$$

In the solution of this equation, the expression e^{mx} cannot be equal to zero, which is an evident conclusion. When the remaining components are set to zero, the equation for $\varphi(x)$ is derived. By applying certain trigonometric transformations, the following equation is obtained.

$$\varphi = A\cos(\lambda x) + B\sin(\lambda x) \tag{18}$$

This equation allows for the determination of the mode shape and frequency of the rod. *A* and *B,* that are unknown integral constants in the given mod shape with Eq. (18) can be found using boundary conditions.

4. Nonlocal Finite Element Formulation for Torsional Vibration

The finite element formulation is based on obtaining approximate functions to get values close to the exact solution. In the exact solution of the torsional vibration problem, the right-hand side of the equation must be equal to zero. However, when attempting to solve the rotation angle θ using an approximate method**,** the right-hand side of the equation is not exactly zero. For this purpose, the weighted residual method will be used to produce finite element formulations. According to the method

$$\mathrm{I} = \int_0^1 wR dx \tag{19}$$

where w is weighting function and R is residual. The average weighted residual for a torsional rod of length L is given as follows

$$\mathrm{I} = \int_0^L w\left[GI_p \frac{\partial^2 \theta}{\partial x^2} + (e_0 a)^2 \rho I_p \frac{\partial^4 \theta}{\partial x^2 \partial t^2} - \rho I_p \frac{\partial^2 \theta}{\partial t^2}\right] dx \tag{20}$$

Eq. (20) can be divided into three integral expressions, which are written as follows:

$$\mathrm{I}_1 = \int_0^L GI_p w \frac{\partial^2 \theta}{\partial x^2} dx = GI_p w \frac{\partial \theta}{\partial x}\bigg|_0^L - \int_0^L GI_p \frac{\partial w}{\partial x}\frac{\partial \theta}{\partial x} dx \tag{21}$$

$$\mathrm{I}_2 = \int_0^L (e_0 a)^2 \rho I_p w \frac{\partial^4 \theta}{\partial x^2 \partial t^2} dx = (e_0 a)^2 \rho I_p w \frac{\partial^3 \theta}{\partial x \partial t^2}\Big|_0^L - \int_0^L (e_0 a)^2 \rho I_p \frac{\partial w}{\partial x} \frac{\partial^3 \theta}{\partial x \partial t^2} dx \tag{22}$$

$$\mathrm{I}_3 = \int_0^L \rho I_p w \frac{\partial^2 \theta}{\partial t^2} dx \tag{23}$$

Substituting Eqs. (21–23) into Eq. (20), the weak formulation of differential equation for nonlocal torsional vibration is obtained as follows:

$$\mathrm{I} = \int_0^L \left[-G I_p \frac{\partial w}{\partial x} \frac{\partial \theta}{\partial x} - (e_0 a)^2 \rho I_p \frac{\partial w}{\partial x} \frac{\partial^3 \theta}{\partial x \partial t^2} - \rho I_p w \frac{\partial^2 \theta}{\partial t^2} \right] dx \tag{24}$$

Under boundary conditions, the average weighted residual for the section rotation is given as follows:

$$\left(G I_p w \frac{\partial \theta}{\partial x} + (e_0 a)^2 \rho I_p w \frac{\partial^3 \theta}{\partial x \partial t^2} \right)\Big|_0^L = 0 \tag{25}$$

The finite element formulation for the rotational freedom at each node of the two-node rod element is as follows:

$$\theta_r = [N_1 \quad N_2] \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} = [N]\{\theta\} \tag{26}$$

where

$$N_1 = \left(1 - \frac{x}{L}\right), \quad N_2 = \frac{x}{L} \tag{27}$$

The matrix $[N]$ contains the shape functions of i and j ends. θ_r is angular rotation. θ_i and θ_j denote the rotational deformation components of i and j ends, respectively. On the other hand

$$w = N^T, \qquad \frac{\partial w}{\partial x} = B^T, \qquad \frac{\partial \theta}{\partial x} = B\theta_r, \quad B = D^k N, \qquad \frac{\partial^2 \theta_r}{\partial t^2} = N\ddot{\theta}_r \tag{28}$$

Here, $D^k = \partial/\partial x$ is kinematic operator, N is shape function vector, and $\ddot{\theta}_r$ is angular acceleration. Eq. (24) can be rewritten as:

$$\int_0^L GI_p(B^TB)\theta_r dx + \int_0^L \rho I_p\{(e_0a)^2(B^TB) + (N^TN)\}\ddot{\theta}_r dx = 0 \tag{29}$$

Eq. (29) in general form is expressed as follows:

$$K\theta_r + (I_c + I_{nl})\ddot{\theta}_r = 0 \tag{30}$$

Where:

$$K = \int_0^L GI_p(B^TB)dx = \int_0^L GI_p \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} [N_1' \quad N_2']dx = \frac{GI_p}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{31}$$

$$I_c = \int_0^L \rho I_p(N^TN)dx = \int_0^L \rho I_p \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} [N_1 \quad N_2]dx = \frac{\rho I_p L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{32}$$

$$\begin{aligned} I_{nl} &= \int_0^L \rho I_p(e_0a)^2(B^TB)dx \\ &= \int_0^L \rho I_p(e_0a)^2 \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} [N_1' \quad N_2']dx = \frac{\rho I_p(e_0a)^2}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{33}$$

Here, K is the torsional stiffness matrix of the finite element, I_c is the classical torsional inertia and I_{nl} is the nonlocal torsional inertia matrix. The FEM describes physical systems as linear equation system and focuses on the algebraic analysis of equation system [31]. The natural frequencies corresponding to the torsional vibrations of the system are obtained by equating the determinant of this linear system of equations to zero.

$$\det([K] - \eta^2[I_r]) = 0, \qquad [I_r] = \sum_{i=1}^{n}([I_c] + [I_{nl}]) \tag{34}$$

Here, η represents the natural frequency of the system, and I_r is the global inertia matrix.

5. Concluding Remarks

In this study, analytical and nonlocal finite element solutions for the atomic size-dependent torsional vibration of circular nanorods are obtained, and these solutions are compared. NL-FEM solutions consistently reveal the effect of the nonlocal parameter in results such as the increase of angular rotation and the decrease of natural frequency. The NL-FEM abandons the differential form and instead applies the original integral nonlocal stress relation which considers the precise nonlocal boundary effect [32]. Therefore, the results are more accurate and sensitive to nonlocal effects. Furthermore, finite element formulation provides effective results in complex loading cases or boundary conditions where analytical methods are limited. On the other hand, in the derivations of analytical formulations the atomic size-effect (e_0a)

affects the rotational mass inertia (ρI_p) parameter. Similarly, in the finite element formulation, the atomic size effect combines with the rotational mass inertia and thus yields an additional mass matrix to the classical theory. The increase in mass matrix indicates that the atomic size effect has a reducing effect on torsion frequencies. This result reveals that the atomic-size effect should be considered in the nano-electro-mechanical system elements under torsional vibration.

Author Contribution

Aleyna Yazıcıoglu: Conducted a literature review, Conceived and designed the analysis, Wrote the paper, Prepared the paper for publication.

Ömer Civalek: Conducted a literature review, Designed and directed the paper, Verified the theories and methods, Prepared the paper for publication

References

[1] Işık, Ç., Nonlocal Finite Element Formulation In Modeling Micro and Nano Scaled Mechanical Systems; PhD thesis, *Akdeniz University*, 2018.

[2] Numanoğlu, H.M. and Civalek, O., On shear-dependent vibration of nano frames. *International Journal of Engineering Science*, 103992, 2024.

[3] Edelen, D.G.B., On nonlocal variational mechanics I. *International Journal of Engineering Science, 7*(3), 269-285, 1969.

[4] Edelen, D.G.B., Nonlocal variations and local invariance of fields. *Modern Analytic and Computation Methods in Science and Mathematics,* 19, 1969.

[5] Eringen, A.C., Nonlocal polar elastic continua. *International Journal of Engineering Science*, 10(1), 1-16, 1972.

[6] Eringen, A.C., Theory of thermomicrofluids. *Journal of Mathematical analysis and Applications*, 38(2), 480-496, 1972.

[7] Eringen, A.C., Theory of nonlocal thermoelasticity. *International Journal of Engineering Science*, 12(12), 1063-1077, 1974.

[8] Eringen, A.C., Plane waves in nonlocal micropolar elasticity. *International Journal of Engineering Science*, 22(8-10), 1113-1121, 1984.

[9] Reddy, J.N., Nonlocal theories for bending, buckling and vibration of beams. *International Journal of Engineering Science*, 45(2-8): 288-307,2007.

[10] Aydoğdu, M., Axial vibration of the nanorods with the nonlocal continuum rod model. *Physica E*, 41(5), 861-864, 2009.

[11] Civalek, O. and Demir, Ç., Bending analysis of microtubules using nonlocal Euler–Bernoulli beam theory. *Applied Mathematical Modelling*, 35(5), 2053-2067, 2011.

[12] Civalek, O., Demir, Ç. and Akgöz, B., Static analysif of single walled carbon nanotubes (SWCNT) based on Eringen's nonlocal elasticity theory. *International Journal of Engineering and Applied Science*, 1(2), 47-56, 2009.

[13] Civalek, O. and Demir, C., Buckling and bending analyses of cantilever carbon nanotubes using the Euler-Bernoulli beam theory based on non-local continuum model; technical note. *Asian Journal of Civil Engineering (Building and Housing)*, 12, 651-661, 2011.

[14] Li, C., Torsional vibration of carbon nanotubes: Comparison of two nonlocal models and a semi-continuum model. *International Journal of Mechanical Sciences*, 82, 25-31, 2014.

[15] Xu, X.J., Zheng, M.L. and Wang, X.C., On vibration of nonlocal rods: Boundary conditions, exact solutions and their asymptotics. *International Journal of Engineering Science*, 119, 217-231, 2017.

[16] Numanoğlu, H. M. and Civalek, O., Novel size-dependent finite element formulation for modal analysis of cracked nanorods. *Materials Today Communications*, 31, 103545, 2022.

[17] Fakher, M. and Hosseini-Hashemi, S., Vibration of two-phase local/nonlocal Timoshenko nanobeams with an efficient shear-locking-free finite-element model and exact solution. *Engineering with Computers*, 1-15, 2022.

[18] Akgöz, B. and Civalek, O., Vibrational characteristics of embedded microbeams lying on a two-parameter elastic foundation in thermal environment. *Composites Part B: Engineering*, 150, 68-77, 2018.

[19] Abouelregal, A.E., Ersoy, H. and Civalek, O., Solution of Moore-Gibson-Thompson equation of an unbounded medium with a cylindrical hole. *Mathematics*, 9, 1536.

[20] Khorasani, M., Soleimani-Javid, Z., Arshid, E., Lampani, L. and Civalek, O., Thermo-elastic buckling of honeycomb micro plates integrated with FG-GNPs reinforced Epoxy skins with stretching effect. *Composite Structures*, 258, 113430.

[21] Mercan, K., Baltacıoglu, A.K. and Civalek, O., Free vibration of laminated and FGM/CNT composites annular thick plates with shear deformation by discrete singular convolution method. *Composite Structures*, 186, 139-153.

[22] Arefi, M., Moghaddam, S.K., Bidgoli, E.M., Kiani, M. and Civalek, O., Analysis of graphene nanoplatelet reinforced cylindrical shell subjected to thermo-mechanical loads. *Composite Structures*, 255, 112924, 2021.

[23] Civalek, O., Akbaş, Ş.D., Akgöz, B. and Dastjerdi, D., Forced vibration analysis of composite beams reinforced by carbon nanotubes. Nanomaterials, 11(3), 571, 2021.

[24] Uzun, B., Yaylı, M. Ö., and Civalek, O., Elastic medium and torsional spring effects on the nonlocal dynamic of functionally graded porous nanotubes. *Archive of Applied Mechanics*, *94*(5), 1291-1311, 2024.

[25] Eringen, A.C., On differential equations of nonlocal elasticity and solutions of screw. dislocation and surface waves. On shear-dependent vibration of nano frames. *Journal of Applied Physics*, 54, 4703-4710, 1983.

[26] Civalek, O. and Akgoz, B., Free vibration analysis of microtubules as cytoskeleton components: nonlocal Euler-Bernoulli beam modeling. *Transaction B: Mechanical Engineering*, 17, 367-375, 2010.

[27] Taati, E., Borjalilou, V., Fallah, F. and Ahmadian, M.T., On size-dependent nonlinear free vibration of carbon nanotube-reinforced beams based on the nonlocal elasticity theory: Perturbation technique. *Mech. Based Des. Struct. Machines,* 5086), 2124–2146, 2022.

[28] Kafkas, U., Dönel Yaylar ile Sabitlenmiş Bir Karbon Nanotüpün Boşluk Oranına Bağlı Serbest Titreşim Analizi. PhD thesis, *Bursa Uludağ University,* 2022.

[29] Reddy, J.N. and Pang, S.D., Nonlocal continuum theories of beams for the analysis of carbon nanotubes. *Journal of applied physics*, 103(2), 2008.

[30] Demir, Ç. and Civalek, O., Torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models. *Applied Mathematical Modelling*, 37(22), 9355-9367, 2013.

[31] Numanoğlu, H.M. and Civalek, O., On the torsional vibration of nanorods surrounded by elastic matrix via nonlocal FEM. *International Journal of Mechanical Sciences*, 161, 105076, 2019.

[32] Lim, C. W., Islam, M. Z., & Zhang, G., A nonlocal finite element method for torsional statics and dynamics of circular nanostructures. *International Journal of Mechanical Sciences*, *94*, 232-243, 2015.